Chaos
An Interdisciplinary Journal of Nonlinear Science

RESEARCH ARTICLE | APRIL 25 2023

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V. Gáspár ✉ ; J. Tóth

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Chaos 33, 043141 (2023)
https://doi.org/10.1063/5.0130349

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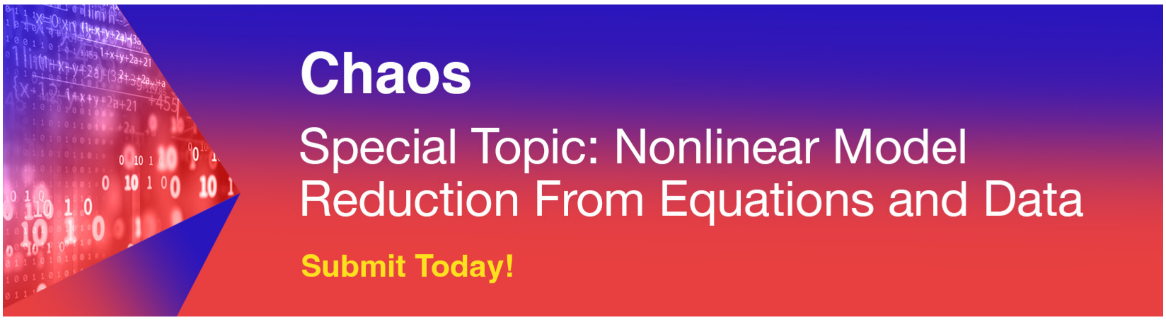

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Cite as: Chaos **33**, 043141 (2023); doi: 10.1063/5.0130349
Submitted: 11 October 2022 · Accepted: 3 April 2023 ·
Published Online: 25 April 2023

V. Gáspár[1,a)] and J. Tóth[2,b)]

AFFILIATIONS

[1] Laboratory of Nonlinear Chemical Dynamics, Institute of Chemistry, Eötvös Loránd University, Pázmány Péter sétány 1/A, H-1117 Budapest, Hungary
[2] Department of Analysis, Institute of Mathematics, Budapest University of Technology and Economics, Műegyetem rkp. 3., H-1111 Budapest, Hungary

[a)] **Author to whom correspondence should be addressed:** vilmos.gaspar@gmail.com
[b)] **Electronic mail:** jtoth@math.bme.hu. **Also at:** Chemical Kinetics Laboratory, Institute of Chemistry, Eötvös Loránd University, Pázmány Péter sétány 1/A, H-1117 Budapest, Hungary.

ABSTRACT

The concept of reaction extent (the progress of a reaction, advancement of the reaction, conversion, etc.) was introduced around 100 years ago. Most of the literature provides a definition for the exceptional case of a single reaction step or gives an implicit definition that cannot be made explicit. There are views that the reaction extent somehow has to tend to 1 when the reaction goes to completion as time tends to infinity. However, there is no agreement on which function should tend to 1. Starting from the standard definition by IUPAC and following the classical works by De Donder, Aris, and Croce, we extend the definition of the reaction extent for an arbitrary number of species and reaction steps. The new general, explicit definition is also valid for non-mass action kinetics. We also studied the mathematical properties (evolution equation, continuity, monotony, differentiability, etc.) of the defined quantity, connecting them to the formalism of modern reaction kinetics. Our approach tries to adhere to the customs of chemists and be mathematically correct simultaneously. To make the exposition easy to understand, we use simple chemical examples and many figures, throughout. We also show how to apply this concept to exotic reactions: reactions with more than one stationary state, oscillatory reactions, and reactions showing chaotic behavior. The main advantage of the new definition of reaction extent is that by knowing the kinetic model of a reacting system one can now calculate not only the time evolution of the concentration of each reacting species but also the number of occurrences of the individual reaction events.

Newton and Leibniz developed calculus (independently) around the turn of the 17th century. Their method of applying ordinary differential equations (ODEs) to predict changes in different physical quantities in time revolutionized classical physics. Interestingly, the first application of calculus to a chemical problem took place much later. Wilhelmy's pioneering research into the kinetics of sucrose inversion has been published in 1850. Although Wilhelmy's groundbreaking work has not been recognized in his lifetime, the science of chemical kinetics later developed into a major branch of physical chemistry. A great practical advantage of applying ODEs to describe the kinetics of chemical processes is that one can predict the concentration of the chemical components as time progresses. Modern numerical methods allow us to solve (integrate) the system of ODEs even for chemical systems that show as complex dynamics as chaos. On the contrary, at present, there exists no method to characterize the extent of each individual reaction in such a complex reaction network. The concept of reaction extent (the progress of a reaction, advancement of the reaction, etc.) was introduced around 100 years ago. However, most of the literature provides a definition for the exceptional case of a single reaction step only. Starting from the standard definition by IUPAC and following the classical works by De Donder, Aris, and Croce, we now extend the definition of the reaction extent for an arbitrary number of species and reaction steps. The main advantage of the new definition is that by knowing the kinetic model of a reacting system we can now calculate not only the time evolution of the concentration of each reacting species but also the number of occurrences of the individual reaction events. This new possibility provides a useful numerical tool for the large community

of researchers who study and analyze complex reaction networks, for example, in combustion chemistry, biochemical systems, and atmospheric chemistry. Our results reveal that there existed a kind of sleeping definition with no use in chemical kinetics, and we show that this should not be the case.

I. INTRODUCTION

The concept of reaction extent (most often denoted by ξ) is almost 100 years old, see De Donder and Van Rysselberghe (1936). Its importance is emphasized by the mere fact that it has been included in the IUPAC Green Book (Mills *et al.*, 1993, p. 43). Two definitions are given that are equivalent in the treated very simple special case of a single reaction step,

$$\sum_{m=1}^{M} \alpha_m \mathrm{X}(m) = \sum_{m=1}^{M} \beta_m \mathrm{X}(m), \tag{1}$$

where M is the number of chemical species $\mathrm{X}(1), \mathrm{X}(2), \ldots, \mathrm{X}(M)$; and the integers α_m and β_m are the corresponding stoichiometric coefficients of the reactant and product species, respectively. The first definition is

$$n_{\mathrm{X}(m)} = n^0_{\mathrm{X}(m)} + \nu_m \xi, \tag{2}$$

where $\mathrm{n}_{\mathrm{X}(m)}$ and $n^0_{\mathrm{X}(m)}$ are the actual and initial quantities (number of moles) of the species $\mathrm{X}(m)$, respectively. The symbol ν_m is the generalized stoichiometric number. It is negative for a reactant and positive for a product species. The second definition is

$$\Delta\xi = \frac{\Delta n_{\mathrm{X}(m)}}{\nu_m} = \frac{n_{\mathrm{X}(m)} - n^0_{\mathrm{X}(m)}}{\nu_m}. \tag{3}$$

A slightly different version is given in Gold *et al.* (1987) and in the electronic version https://goldbook.iupac.org/terms/view/E02283 called IUPAC Gold Book (McNaught and Wilkinson, 1997),

$$\mathrm{d}\xi = \frac{\mathrm{d}n_{\mathrm{X}(m)}}{\nu_m}. \tag{4}$$

The above-cited definitions have been summarized in the book by Stepanov *et al.* (1976). The authors also give a good introduction to the methods of linear algebra applied in Reaction Kinetics.

With an eye on the applicability of the concept in modern formal reaction kinetics (or, chemical reaction network theory) as exposed by Tóth *et al.* (2018) and Feinberg (2019) the following points seem crucial:

1. Starting from the original definition by De Donder and Van Rysselberghe (1936), see also Bowen (1968), we extend the definition to an **arbitrary** number of reaction steps.
2. We do not restrict ourselves to reversible steps.
3. We do not require linear independence of the reaction steps.
4. We do not "order the steps to one side," which would result in hiding the difference between the steps like $\mathrm{X} \longrightarrow \mathrm{Y}$ and $\mathrm{X} + \mathrm{Y} \longrightarrow 2\mathrm{Y}$.
5. We do not consider and take into account the atomic (or molecular) structure of the species.
6. We do not use differentials when introducing the concept; cf. Truesdell (1984, p. 61).
7. We shall give an explicit definition more similar to Eq. (3) rather than to Eq. (2).
8. We take into consideration the volume of the reacting mixture to be able to calculate the number of individual reaction events.

The structure of our paper is as follows. Section II introduces the concept for reaction networks of arbitrary complexity: for any number of reaction steps and species, mass action kinetics is not assumed. As it is a usual requirement that the reaction extent tends to 1 when "the reaction tends to its end," we try to find quantities derived from our reaction extent having this property in Sec. III. It will turn out in many examples that the reaction extents do not tend to 1 in any sense. We show, however, that they contain quite relevant information about the time evolution of the reactions: they measure (or give) the number of occurrences of the individual reaction events. These examples will also reflect the fact that the reaction events do not cease during equilibrium, and this can be seen without referring to fluctuations.

At the closure of our paper, we show applications of the concept to more complicated cases: those with multiple stationary states, oscillation, and chaos. In this part, first, we analyze the classic multistationary example by Horn and Jackson (1972). As to oscillatory reactions, we start with the irreversible Lotka–Volterra reaction, and we also study the reversible Lotka–Volterra reaction in both the detailed balanced and not detailed balanced cases. Our following oscillatory example will be an experimental system studied by Rábai (1998). As a chaotic example, we shall take a slightly modified version of this oscillatory system.

Discussion of the conclusions comes last. The proofs of statements and theorems are relegated to an Appendix so as to improve the logical flow of the manuscript without getting side-tracked. The applied notations and mathematical symbols are collected in Tables II and III in the Appendix. See the supplementary material for all calculations and drawings made by using the Wolfram language.

II. THE CONCEPT OF REACTION EXTENT

Starting from the classical works by De Donder and Van Rysselberghe (1936), Aris (1965b), and Croce (2002), and relying on the consensus of the chemists' community as formulated by Laidler (1996), our aim is to present a more general treatment than any of the definitions introduced and applied earlier.

A. Motivation and fundamental definitions

We are going to use the following concepts:

1. Fundamental notations and definitions: The framework

Following the books by Tóth *et al.* (2018) and Feinberg (2019), we consider a **complex chemical reaction**, simply **reaction**, or **reaction network** as a set consisting of **reaction steps** as follows:

$$\left\{ \sum_{m=1}^{M} \alpha_{m,r} \mathrm{X}(m) \longrightarrow \sum_{m=1}^{M} \beta_{m,r} \mathrm{X}(m) \quad (r = 1, 2, \ldots, R) \right\}, \tag{5}$$

where

1. the chemical **species** are X(1), X(2), …, X(M)—take note that their quantities $N_{\mathrm{X}(m)}$ or N_m will be applied interchangeably;
2. the **reaction steps** are numbered from 1 to R;
3. here M and R are positive integers;
4. $\boldsymbol{\alpha} := [\alpha_{m,r}]$ and $\boldsymbol{\beta} := [\beta_{m,r}]$ are $M \times R$ matrices of non-negative integer components called **stoichiometric coefficients**, with the properties that all the species take part in at least one reaction step ($\forall m \exists r : \beta_{m,r} \neq \alpha_{m,r}$), and all the reaction steps do have some effect ($\forall r \exists m : \beta_{m,r} \neq \alpha_{m,r}$), and
5. $\boldsymbol{\gamma} := \boldsymbol{\beta} - \boldsymbol{\alpha}$ is the **stoichiometric matrix** of **stoichiometric numbers**, finally
6. the formal linear combinations on the two sides of the reaction steps in Eq. (5) are called complexes, and their **length** is the sum of their stoichiometric coefficients.

Instead of Eq. (5) some authors prefer writing this

$$\sum_{m=1}^{M} \gamma_{m,r}\mathrm{X}(m) = 0 \quad (r = 1, 2, \ldots, R). \tag{6}$$

This formulation immediately excludes reaction steps like $\mathrm{X} + \mathrm{Y} \longrightarrow 2\mathrm{Y}$ (used, e.g., to describe a step in the Lotka–Volterra reaction) or reduces it to $\mathrm{X} \longrightarrow \mathrm{Y}$, changing the stoichiometric coefficients used to formulate mass-action type kinetics. Similarly, an autocatalytic step that may be worth studying, see, e.g., Aris (1965a, pp. 63 and 66), like $\mathrm{X} \longrightarrow 2\mathrm{X}$ appears oversimplified as $0 \longrightarrow \mathrm{X}$. Another possibility is to exclude the **empty complex**, implying involuntarily that we get rid of the possibility to simply represent in- and outflow with $0 \longrightarrow \mathrm{X}$ and $\mathrm{X} \longrightarrow 0$, respectively. These last two examples evidently mean mass-creation and mass-destruction. If one does not like these kinds of representations of reaction steps, one should explicitly say that one is only interested in mass-conserving reactions. Sometimes, mass creation and mass destruction are slightly less obvious than above, see the reaction network

$$\mathrm{X} \longrightarrow \mathrm{Y} + \mathrm{U}, \quad \mathrm{Z} \longrightarrow \mathrm{X} + \mathrm{U}, \quad \mathrm{Z} \longrightarrow \mathrm{U},$$

which is mass-producing.

It may happen that one would like to exclude reaction steps with more than two particles on the left side, such as

$$2\mathrm{MnO}_4^- + 6\mathrm{H}^+ + 5\mathrm{H}_2\mathrm{C}_2\mathrm{O}_4 = 2\mathrm{Mn}^{2+} + 8\mathrm{H}_2\mathrm{O} + 10\mathrm{CO}_2.$$

Such steps do occur, e.g., here (Kovács *et al.*, 2004, p. 1236) when dealing with **overall reactions**. The theory and applications of decomposition of overall reactions into elementary steps (Kovács *et al.*, 2004 and Papp and Vizvári, 2006) would have been impossible without the framework of formal reaction kinetics. Someone may be interested in complex chemical reactions consisting of reversible steps only. Then, they have to write down all the forward and the corresponding backward reaction steps. *Let us emphasize that we do not make such assumptions as detailed above; we are merely pointing out some possible assumptions people make. Instead, the assumptions we apply throughout this work will be made explicit.*

Taking into consideration restrictions of the above kind usually does not make the mathematical treatment easier. Sometimes, it needs hard work to figure out how they can be checked, as it is in the case of mass conservation of models containing species without atomic structure (Deák *et al.*, 1992 and Tóth *et al.*, 2018), or in relation to the existence of oscillatory reactions: Póta (1983). To sum up: an author has the right to make any restriction thought to be chemically important, but these restrictions should be declared at the outset. Finally, we mention our main assumption: all the steps in Eq. (5) are really present, i.e., they proceed with a positive rate whenever the species on the left side are present.

We now provide a simple example to **motivate** the formulation of the new, general, and explicit Definition 1 of the reaction extent (to be shown later) that is also consistent with the standard definition by IUPAC.

2. A simple example

Let us take an example that may be deemed chemically oversimplified but not too trivial, still simple enough so as not to be lost in the details. Assume that water formation follows the reversible reaction step $2\mathrm{H}_2 + \mathrm{O}_2 \rightleftharpoons 2\mathrm{H}_2\mathrm{O}$. This means that we do not take into consideration the realistic details of water formation. Let the forward step be represented in a more abstract way: $2\mathrm{X} + \mathrm{Y} \longrightarrow 2\mathrm{Z}$. Here, the atomic structure of the species is also disregarded.

The number of species, denoted as above by M, is 3, and the number of (irreversible) reaction steps, denoted as above by R, is 1. The 3×1 stoichiometric matrix $\boldsymbol{\gamma}$ consisting of the stoichiometric numbers is $\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$. In case this step occurs five times, the vector of the **number of individual species** will change as follows:

$$\begin{bmatrix} N_\mathrm{X} - N_\mathrm{X}^0 \\ N_\mathrm{Y} - N_\mathrm{Y}^0 \\ N_\mathrm{Z} - N_\mathrm{Z}^0 \end{bmatrix} = 5 \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix},$$

where N_X^0 is the number of molecules of species X at the beginning, N_X is the number of molecules of species X after five reaction events, and so on. If one considers the reversible reaction $2\mathrm{X} + \mathrm{Y} \rightleftharpoons 2\mathrm{Z}$, then the 3×2 stoichiometric matrix $\boldsymbol{\gamma}$ consisting of the stoichiometric numbers is as follows:

$$\begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix}.$$

Assuming that the backward reaction step takes place three times then the total change is

$$\begin{bmatrix} N_\mathrm{X} \\ N_\mathrm{Y} \\ N_\mathrm{Z} \end{bmatrix} - \begin{bmatrix} N_\mathrm{X}^0 \\ N_\mathrm{Y}^0 \\ N_\mathrm{Z}^0 \end{bmatrix} = 5 \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}. \tag{7}$$

Note that both the number of molecules and the number of the occurrence of reaction events are positive integers.

Equation (2.2) of Kurtz (1972) is of the same form as our Eq. (7). Kurtz is interested mainly in reversible and detailed balanced complex chemical reactions, and, more importantly, in the relationship of their deterministic and stochastic models. This is the reason why he formulates his Eq. (2.2) for the slightly restricted case only. As to the relationship between discrete and continuous descriptions, we follow here more or less Kurtz (1972) and Tóth *et al.* (2018). We cannot rely on a discrete state deterministic model of reaction kinetics—that would be desirable—because such a model does not exist as far as we know.

3. The general case

Before providing general definitions, we mention that Dumon *et al.* (1993) formulated a series of requirements that—according to them—should be obeyed by a well-defined reaction extent. Unfortunately, we are unable to accept most of these requirements. Let us mention only one: the reaction extent should be independent of the choice of stoichiometric coefficients (invariant under multiplication), i.e., it should have the same value for the reaction $2H_2 + O_2 \longrightarrow 2H_2O$ and for the reaction $H_2 + \frac{1}{2}O_2 \longrightarrow H_2O$. Our point of view is that the reaction extent is strongly connected to kinetics, and it is not a tool to describe stoichiometry as some other authors also think [see, e.g., Garst (1974)]. The only requirement that we accept will be mentioned later, in the discussion of Definition 1.

We assume throughout that the volume (V) is constant, and one can write the generalized form of Eq. (7) as

$$\begin{bmatrix} N_1 \\ N_2 \\ \dots \\ N_{\mathrm{M}} \end{bmatrix} - \begin{bmatrix} N_1^0 \\ N_2^0 \\ \dots \\ N_{\mathrm{M}}^0 \end{bmatrix} = \sum_{r=1}^{R} \begin{bmatrix} \gamma_{1,r} \\ \gamma_{2,r} \\ \dots \\ \gamma_{M,r} \end{bmatrix} W_r,$$

or shortly

$$\boldsymbol{N} - \boldsymbol{N}^0 = \boldsymbol{\gamma} \boldsymbol{W},$$

where component W_r of the vector $\boldsymbol{W} = \begin{bmatrix} W_1 & W_2 & \dots & W_R \end{bmatrix}^\top$ gives the number of occurrences of the rth reaction step. Note that we do not speak about **infinitesimal** changes.

With a slight abuse of notation, let $\boldsymbol{W}(t)$ be the vector of the number of occurrences of reaction events, a step function in the interval $[0, t]$. Then,

$$\boldsymbol{N}(t) - \boldsymbol{N}^0 = \boldsymbol{\gamma} \boldsymbol{W}(t),$$

or turning to moles

$$\boldsymbol{n}(t) - \boldsymbol{n}^0 = \frac{\boldsymbol{N}(t) - \boldsymbol{N}^0}{L} = \boldsymbol{\gamma} \frac{\boldsymbol{W}(t)}{L} = \boldsymbol{\gamma} \boldsymbol{\xi}(t), \qquad (8)$$

where L **is the Avogadro constant** having the unit mol^{-1}, and

$$\boldsymbol{n}(t) := \frac{\boldsymbol{N}(t)}{L}, \ \boldsymbol{n}^0 := \frac{\boldsymbol{N}^0}{L}, \ \boldsymbol{\xi}(t) := \frac{\boldsymbol{W}(t)}{L}.$$

Here, we had to choose the less often used notation L (https://goldbook.iupac.org/terms/view/A00543) to avoid mixing up with other notations.

The relationship Eq. (8) can be expressed in concentrations as

$$\boldsymbol{c}(t) - \boldsymbol{c}^0 = \frac{\boldsymbol{n}(t) - \boldsymbol{n}^0}{V} = \frac{\boldsymbol{\gamma}}{V} \boldsymbol{\xi}(t), \qquad (9)$$

where $V \in \mathbb{R}^+$, the volume of the reaction vessel is assumed to be constant, $\boldsymbol{c}(t) := \frac{\boldsymbol{n}(t)}{V}$ and $\boldsymbol{c}^0 := \frac{\boldsymbol{n}^0}{V}$. The component $c_{\mathrm{X}(m)}$ or c_m of $\boldsymbol{c}$ is traditionally denoted in chemical textbooks as $[\mathrm{X}(m)]$, see, e.g., Pilling and Seakins (1995, Sec. 1.2).

The concentration in Eq. (9) is again a step function; however, if the number of particles (molecules, radicals, electrons, etc.) is very large, as very often it is, it may be considered to be a continuous, even a differentiable function. Remember that the components of $\boldsymbol{\xi}(t)$ have the dimension of the amount of substance, measured in moles.

Let us now give a general, formal, and **explicit** definition of reaction extent valid **for an arbitrary number of species and reaction steps, and not restricted to mass action type kinetics.** [Few qualitative—mainly technical—restrictions are usually made on the function **rate**, see Volpert and Hudyaev (1985), Tóth *et al.* (2018), and Feinberg (2019), but we now mention the continuous differentiability only.]

We start with rewriting the induced kinetic differential equation,

$$\dot{\boldsymbol{c}}(t) = \boldsymbol{\gamma}\,\mathbf{rate}(\boldsymbol{c}(t)), \qquad (10)$$

together with the initial condition $\boldsymbol{c}(0) = \boldsymbol{c}^0$ into an (equivalent) integral equation,

$$\boldsymbol{c}(t) - \boldsymbol{c}^0 = \boldsymbol{\gamma} \int_0^t \mathbf{rate}(\boldsymbol{c}(\bar{t}))\,\mathrm{d}\bar{t}. \qquad (11)$$

The component $rate_r$ of the vector **rate** provides the reaction rate of the rth reaction step. Note that in the mass action case, Eq. (10) in coordinates specializes into

$$\dot{c}_m(t) = \sum_{r=1}^{R} \gamma_{m,r} k_r \prod_{p=1}^{M} c_p^{\alpha_{p,r}} \quad (m = 1, 2, \dots, M),$$

where $\boldsymbol{k}_r$ is the (positive) reaction rate coefficient of the rth reaction step. (We prefer using the expression **reaction rate coefficients** to **reaction rate constants**, as these numbers do depend on many factors—except species concentrations.)

In accordance with what has been said up to now, we can introduce the explicit definition of reaction extent by comparing Eqs. (9) and (11). However, we emphasize that this procedure serves **as a motivation and not as a direct derivation.**

Definition 1. The **reaction extent** of a complex chemical reaction or reaction network defined by Eq. (5) is the scalar variable, vector-valued function given by the formula

$$\boxed{\boldsymbol{\xi}(t) := V \int_0^t \mathbf{rate}(\boldsymbol{c}(\bar{t}))\,\mathrm{d}\bar{t}.} \qquad (12)$$

Its time derivative $\dot{\boldsymbol{\xi}}(t) = V\mathbf{rate}(\boldsymbol{c}(t))$ is usually called the **rate of conversion** or **reaction flux**, see, e.g., Polettini and Esposito (2014).

Note that Eq. (12) shows that the reaction extent, in general, depends on the whole history (past and present) of the vector of concentrations, as if it had a **memory**.

Definition 1 of the reaction extent has been derived from the number of reaction events in order to reveal its connection to changes in the concentrations. Assuming here also that V is constant, one can formulate the following trivial (equivalent) consequences of the definition:

$$\dot{\boldsymbol{n}} = \boldsymbol{\gamma}\dot{\boldsymbol{\xi}}, \quad \dot{\boldsymbol{c}} = \frac{1}{V}\boldsymbol{\gamma}\dot{\boldsymbol{\xi}}, \quad \boldsymbol{c} = \boldsymbol{c}^0 + \boldsymbol{\gamma}\frac{\boldsymbol{\xi}}{V}, \tag{13}$$

mentioned also by Laidler (1996), sometimes as definitions, sometimes as statements. Another consequence we shall most often use is as follows:

$$\dot{\boldsymbol{\xi}} = V\mathbf{rate}(\boldsymbol{c}^0 + \boldsymbol{\gamma}\boldsymbol{\xi}/V). \tag{14}$$

This and the relations (13) show convincingly that the given definition of the reaction extent will be useful. Let us also emphasize that $\dot{\boldsymbol{n}} = \boldsymbol{\gamma}\dot{\boldsymbol{\xi}}$ gives $\mathrm{d}\boldsymbol{n} = \boldsymbol{\gamma}\,\mathrm{d}\boldsymbol{\xi}$, the standard definition by IUPAC for the simplest case.

Note that neither the rate of the reaction: $\mathbf{rate}(\boldsymbol{c}(t)) = \frac{\dot{\boldsymbol{\xi}}(t)}{V}$, nor the reaction extent $\boldsymbol{\xi}$, nor the rate of conversion $\dot{\boldsymbol{\xi}}$ depends on the stoichiometric matrix $\boldsymbol{\gamma}$, thereby this one of the requirements formulated by Dumon *et al.* (1993) is fulfilled.

What is wrong with the almost ubiquitous implicit "definition" Eq. (9)? We show an example to enlighten this.

Example 1. Consider the reaction steps

$$\mathrm{X} \xrightarrow{k_1} \mathrm{Y}, \quad \mathrm{X} + \mathrm{Y} \xrightarrow{k_2} 2\mathrm{Y},$$

expressing the fact that X is transformed into Y directly and also via autocatalysis. Although the reaction steps

$$\mathrm{X} \xrightarrow{k_1} \mathrm{Y} + \mathrm{P}, \quad \mathrm{X} + \mathrm{Y} \xrightarrow{k_2} 2\mathrm{Y} + \mathrm{P}$$

with the external species P is a more realistic description of genuine chemical reactions, e.g., the acid autocatalysis in ester hydrolysis: Ostwald (1884) and Bánsági and Taylor (2017), they lead to the same kinetic differential equations for X and Y. Therefore, we shall analyze the simpler scheme. Now the stoichiometric matrix $\boldsymbol{\gamma}$ is as follows:

$$\boldsymbol{\gamma} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Then, Eq. (11) specializes into

$$\begin{aligned} c_{\mathrm{X}}(t) - c_{\mathrm{X}}(0) &= -\int_0^t k_1 c_{\mathrm{X}}(\bar{t})\,\mathrm{d}\bar{t} - \int_0^t k_2 c_{\mathrm{X}}(\bar{t}) c_{\mathrm{Y}}(\bar{t})\,\mathrm{d}\bar{t} \\ &= -\frac{\xi_1(t) + \xi_2(t)}{V}, \end{aligned}$$

$$c_{\mathrm{Y}}(t) - c_{\mathrm{Y}}(0) = \int_0^t k_1 c_{\mathrm{X}}(\bar{t})\,\mathrm{d}\bar{t} + \int_0^t k_2 c_{\mathrm{X}}(\bar{t}) c_{\mathrm{Y}}(\bar{t})\,\mathrm{d}\bar{t} = \frac{\xi_1(t) + \xi_2(t)}{V}.$$

These two relations do not determine ξ_1 and ξ_2 individually but only their sum (even if one utilizes $c_{\mathrm{Y}}(t) = c_{\mathrm{X}}(0) + c_{\mathrm{Y}}(0) - c_{\mathrm{X}}(t)$). The problem originates from the fact that the reaction steps are not linearly independent as reflected in the singularity of the matrix $\boldsymbol{\gamma}$.

If the reaction steps of a complex chemical reaction are independent, the situation is better.

Example 2. In some special cases, there is a way of making the "definition" Eq. (9) into a real, explicit definition. Assume that $R \leq M$ and that the stoichiometric matrix $\boldsymbol{\gamma}$ is of the full rank, i.e., the reaction steps are independent. Then, one can rewrite Eq. (9) in two steps as follows:

$$\boldsymbol{\gamma}^\top(\boldsymbol{c}(t) - \boldsymbol{c}^0) = \frac{1}{V}\boldsymbol{\gamma}^\top\boldsymbol{\gamma}\boldsymbol{\xi}(t),$$

$$\boldsymbol{\xi}(t) = V(\boldsymbol{\gamma}^\top\boldsymbol{\gamma})^{-1}\boldsymbol{\gamma}^\top(\boldsymbol{c}(t) - \boldsymbol{c}^0). \tag{15}$$

Now one can accept Eq. (15) as a definition for the reaction extent. Nevertheless, in this special case, Eq. (11) implies

$$\boldsymbol{\gamma}^\top(\boldsymbol{c}(t) - \boldsymbol{c}^0) = \boldsymbol{\gamma}^\top\boldsymbol{\gamma}\int_0^t \mathbf{rate}(\boldsymbol{c}(\bar{t}))\,\mathrm{d}\bar{t}$$

and

$$(\boldsymbol{\gamma}^\top\boldsymbol{\gamma})^{-1}\boldsymbol{\gamma}^\top(\boldsymbol{c}(t) - \boldsymbol{c}^0) = \frac{1}{V}\boldsymbol{\xi}(t) = \int_0^t \mathbf{rate}(\boldsymbol{c}(\bar{t}))\,\mathrm{d}\bar{t}.$$

Thus, this definition is the same as the one in Eq. (12). This derivation can always be done if $R = 1$. On the contrary, for example, in the case of combustion reactions, Law's law (Law, 2007, p. 11) states that $R \approx 5M$.

Note also that Eq. (15) shows the following: in these cases, i.e., when the stoichiometric matrix is of the full rank—as opposed to the general case—the reaction extents do not depend on the whole history of the concentration vector, it only depends on its instantaneous value.

Let us make a trivial remark on the independence of reaction steps. If the complex chemical reaction consists of a single irreversible step, then the reaction steps(!) are independent. If any of the reaction steps are reversible, then the reaction steps are not independent.

B. Properties of the reaction extent

The usual assumptions on the vector-valued function **rate** are as follows, see Volpert and Hudyaev (1985) and Tóth *et al.* (2018).

1. All of its components are continuously differentiable functions defined on $(\mathbb{R}_0^+)^M$ taking only non-negative values. This is usual, e.g., in the case of mass action kinetics, but also in the case when the reaction rates are rational functions as in the case of Michaelis–Menten or Holling type kinetics, see, e.g., Laidler (1996), Kiss and Tóth (2009), and Polcz *et al.* (2018a, 2018b).
2. The value of $rate_r(\boldsymbol{c})$ is zero if and only if some of the species needed for the rth reaction step is missing, i.e., for some m : $\alpha_{m,r} > 0$ and $c_m = 0$ (Volpert and Hudyaev, 1985, see Condition 1 in p. 613). We shall say in this case that reaction step r **cannot start** from the concentration vector $\boldsymbol{c}$.

The second assumption implies—even without restriction to the mass action type kinetics—that $rate_r(\boldsymbol{c}) > 0$ if all the necessary species (**reactants**) are present initially: $\alpha_{m,r} > 0 \Longrightarrow c_m > 0$.

Let us sum up the relevant qualitative characteristics of the reaction extent. Remember that the proof can be found in the Appendix.

Theorem 1.

1. *The domain of the function* $t \mapsto \boldsymbol{\xi}(t)$ *is the same as that of* $\boldsymbol{c}$.
2. *Both* $\boldsymbol{c}$ *and* $\boldsymbol{\xi} \in C^2(J, \mathbb{R}^R)$; *with some open interval* $J \subset \mathbb{R}$ *such that* $0 \in J$.

3. *$\boldsymbol{\xi}$ obeys the following initial value problem:*

$$\dot{\boldsymbol{\xi}}(t) = V\mathbf{rate}\left(\boldsymbol{c}^0 + \frac{1}{V}\boldsymbol{\gamma}\boldsymbol{\xi}(t)\right), \quad \boldsymbol{\xi}(0) = \mathbf{0}. \tag{16}$$

4. *At $t = 0$ the velocity vector $\dot{\boldsymbol{\xi}}(0)$ of the reaction extent (also called the **rate of conversion**) points into the closed first orthant, and this property is kept for all times in the domain of the function $t \mapsto \boldsymbol{\xi}(t)$.*
5. *The components of $\boldsymbol{\xi}$ are either positive, strictly monotonically increasing functions or constant zero. If for some positive time t we find that $\xi_r(t) = 0$ then, obviously, the reaction step r did not start at all at the beginning.*

Let us make a few remarks:

- The last property (positivity) mentioned in the theorem can be realized with $\lim_{t\to+\infty}\xi(t) = +\infty$ (the simplest example for this being X $\longrightarrow$ 2X, $c_X^0 > 0$), or with a finite positive value of $\lim_{t\to+\infty}\xi(t)$, see the example X $\longrightarrow$ Y $\longrightarrow$ Z below.
- Equation (16) shows that we would have got simpler formulas if we used $\frac{\xi(t)}{V}$ as proposed by Aris (1965a p. 44), but this form is valid only if V is constant.
- In the mass action case, both $\boldsymbol{c}$ and $\boldsymbol{\xi}$ are infinitely many times differentiable.
- If one uses kinetics different from the mass action type not fulfilling assumption 1 or 2, or—as Póta (2016) has shown—if one applies an approximation, then it may happen that some of the initially positive concentrations turn to zero.

In order to proceed, we need to make a technical remark on the figures shown hereinafter. We label the first axis (usually, horizontal) in the figures with t/s, where s is the time unit second. Labels of other axes are formed in a similar way. With this procedure, we want to emphasize that the figures show the relationship between pure numbers and not between physical quantities.

The condition in part 5 of Theorem 1 is only necessary but not sufficient as the example below shows.

Example 3. Let us start the consecutive reaction X $\xrightarrow{k_1}$ Y $\xrightarrow{k_2}$ Z from the vector of the initial concentrations: $\begin{bmatrix} c_X^0 & 0 & 0\end{bmatrix}^\top$, and suppose $k_1 \neq k_2$. Although the second step cannot start at the beginning, the second reaction extent is positive for all positive times as the solution of the evolution equations,

$$\dot{\xi}_1 = Vk_1\left(c_X^0 - \frac{\xi_1}{V}\right), \quad \dot{\xi}_2 = Vk_2\left(0 + \frac{\xi_1 - \xi_2}{V}\right), \tag{17}$$

are as follows:

$$\xi_1(t) = Vc_X^0(1 - e^{-k_1 t}),$$

$$\xi_2(t) = \frac{Vc_X^0}{k_2 - k_1}\left(k_2(1 - e^{-k_1 t}) - k_1(1 - e^{-k_2 t})\right).$$

Positivity also follows without any calculations from the fact that the velocity vector of the differential equations in Eq. (17) points inward, into the relative interior of the first quadrant, or using the fact that Eq. (17) are also kinetic-type differential equations.

Note that $\lim_{t\to+\infty}\xi_1(t) = \lim_{t\to+\infty}\xi_2(t) = Vc_X^0$. It means that the number of occurrences of the reaction events for both reactions, and thus the reaction extents, are exactly the same at the end of the whole process. Moreover, it does not depend on the reaction rate coefficients (!).

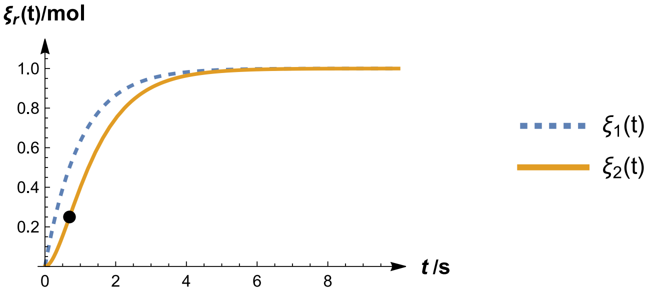

FIG. 1. Reaction extents in the consecutive reaction X $\xrightarrow{k_1}$ Y $\xrightarrow{k_2}$ Z when $k_1 = 1\,\mathrm{s}^{-1}$, $k_2 = 2\,\mathrm{s}^{-1}$, $c_X^0 = 1$ mol dm^{-3}, $c_Y^0 = c_Z^0 = 0$ mol dm^{-3}, and $V = 1$ dm^3. Limit 1 here is only a consequence of the choice of parameters. The large dot is the inflection point of the curve ξ_2/ mol.

Easy calculations show the following facts. The function ξ_2/ mol in Fig. 1 has an inflection point because its second derivative is zero at some positive time t_{infl} for all choices of the reaction rate coefficients, and the third derivative is not zero at t_{infl}. The function ξ_1 / mol in Fig. 1 is concave down no matter what the reaction rate coefficients are.

To characterize the convexity of the reaction extents in the general case is an open problem. One should take into consideration that although in the practically interesting cases the number of equations in Eq. (16) is larger than those in Eq. (10), that is, $R > M$, the equations for the reaction extents are of a simpler structure.

Monotonicity mentioned in Theorem 1 implies that all the components of the reaction extent do have a finite or infinite limit as t tends to $\sup(J) =: t^*$, where $J := \mathrm{Dom}(\boldsymbol{c})$ and t^* is a finite or infinite time. It is an interesting open question: when does a coordinate of the reaction extent vector tend to infinity?

As the **emphatic** closure of this series of remarks, we mention that the strictly monotonic increase of the number of the occurrence of reaction events shows that the reaction events never stop, see Fig. 2. This important fact is independent of the form of kinetics, and it is a property of the deterministic models of reaction kinetics. This sheds light on the meaning of **dynamic equilibrium** as generally taught. Note that **no reference to thermodynamics or statistical physics** has been invoked here; analyzing the connections are left to the reader.

Example 4. In case when the domain of the function $t \mapsto \boldsymbol{c}(t)$ is a proper subset of the non-negative real numbers, $\boldsymbol{\xi}$ has the same property. Let us consider the induced kinetic differential equation $\dot{c} = kc^2$ of the quadratic autocatalytic reaction 2X $\xrightarrow{k}$ 3X with the initial condition $c(0) = c^0 > 0$. Then, $c(t) = \frac{c^0}{1-kc^0t}$, where $t \in \left[0, \frac{1}{kc^0}\right[\subsetneq [0, +\infty[$. Now Eq. (16) specializes into $\dot{\xi} = Vk(c^0 + \xi/V)^2$, $\xi(0) = 0$ having the solution $\xi(t) = V(c^0)^2\frac{kt}{1-kc^0t}$, where $t \in \left[0, \frac{1}{kc^0}\right[$; thus, ξ blows up at the same time $\left(t^* := \frac{1}{kc^0}\right)$ when c does. Up to the blow-up, the reaction events occur infinitely many times: $\lim_{t\to+t^*}\xi(t) = +\infty$. Definitions and a few

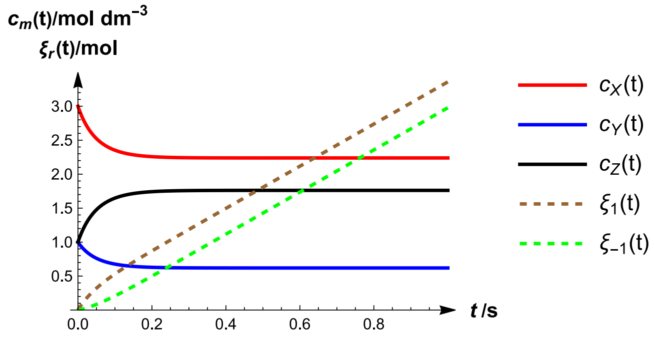

FIG. 2. While the concentrations tend to and are becoming very close to the equilibrium values, the reaction extents tend to infinity in a monotonically increasing way in the reaction: $2X + Y \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2Z$ when $c_X^0 = 3.0$ mol dm^{-3}, $c_Y^0 = 1.0$ mol dm^{-3}, $c_Z^0 = 1.0$ mol dm^{-3}, $k_1 = 1.0\,\mathrm{dm}^6\,\mathrm{mol}^{-2}\,\mathrm{s}^{-1}$, $k_{-1} = 1.0\,\mathrm{dm}^3\,\mathrm{mol}^{-1}\,\mathrm{s}^{-1}$, and $V = 1\,\mathrm{dm}^3$.

statements about blow-up in kinetic differential equations are given in the works by Csikja and Tóth (2007) and Csikja *et al.* (2022).

III. WHAT IS IT THAT TENDS TO 1?

Our interest up to this point was the number of occurrences of reaction events. However, many authors think it is useful and visually attractive that the "reaction extent tends to 1 when the reaction tends to its end," see e.g., (Glasser, 2016, Fig. 1). Peckham (2001) and Borge (2015) also argue for $\xi_r \in [0, 1]$.

Another approach is given by Dumon *et al.* (1993) and Moretti (2015), and others via introducing the **reaction advancement ratio** $\frac{\xi}{\xi_{\max}}$, and stating that this ratio is always between 0 and 1. Peckham (2001) noticed that Atkins (1994, pp. 272–276) shows a figure of the free energy G of the reacting system vs ξ, where the first axis is labeled from zero to one. However, in the next edition (Atkins, 1998, pp. 216–217), the graph has been changed, and it now shows the first axis without 1 as an upper bound. Being loyal to the usual belief (Treptow, 1996 and Peckham, 2001), we are looking for quantities (pure numbers) tending to 1 as e.g., $t \to +\infty$. Different kinds of scaling will help to find such quantities.

A. Scaling by the initial concentration: One reaction step

Now we are descending from the height of generality by considering a single irreversible reaction step ($R = 1$), assuming that the kinetics is of the mass action type. Thus, the reaction is

$$\sum_{m=1}^{M} \alpha_m X(m) \xrightarrow{k} \sum_{m=1}^{M} \beta_m X(m). \tag{18}$$

Therefore, one has the following initial value problem for the reaction extent

$$\dot{\xi} = Vk \left(\begin{bmatrix} c_1^0 \\ c_2^0 \\ \cdots \\ c_M^0 \end{bmatrix} + \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \cdots \\ \gamma_M \end{bmatrix} \frac{\xi}{V} \right)^{\boldsymbol{\alpha}} = Vk \prod_{m=1}^{M} \left(c_m^0 + \gamma_m \frac{\xi}{V} \right)^{\alpha_m},$$

$$\xi(0) = 0, \tag{19}$$

with $\gamma_m := \beta_m - \alpha_m$.

Theorem 2.

1. *If the reaction in Eq. (18) cannot start, then $\xi(t) = 0$ for all nonnegative real times t:*
$$\exists m : (\alpha_m \neq 0 \,\&\, c_m^0 = 0) \Longrightarrow \forall t \in \mathbb{R}_0^+ : \xi(t) = 0. \tag{20}$$
2. *If the reaction in Eq. (18) does start and all the species are produced (i.e., for all m : $\gamma_m \geq 0$ and $\sum_{m=1}^{M} \gamma_m > 0$), then $\xi(t)$ tends to infinity (blow-up included):*
$$\forall m : \gamma_m := \beta_m - \alpha_m > 0 \Longrightarrow \lim_{t \to t^*} \xi(t) = +\infty, \tag{21}$$
where $t^* := \sup(J)$ with $J := \mathrm{Dom}(\xi)$.
3. *If some of the species is consumed, that is, $\exists m : \gamma_m < 0$, then*
$$\lim_{t \to +\infty} \xi(t) = \min \left\{ -\frac{V c_m^0}{\gamma_m}; \gamma_m < 0 \right\}. \tag{22}$$

Figure 22 of the Appendix shows the phase diagram of the three different possibilities.

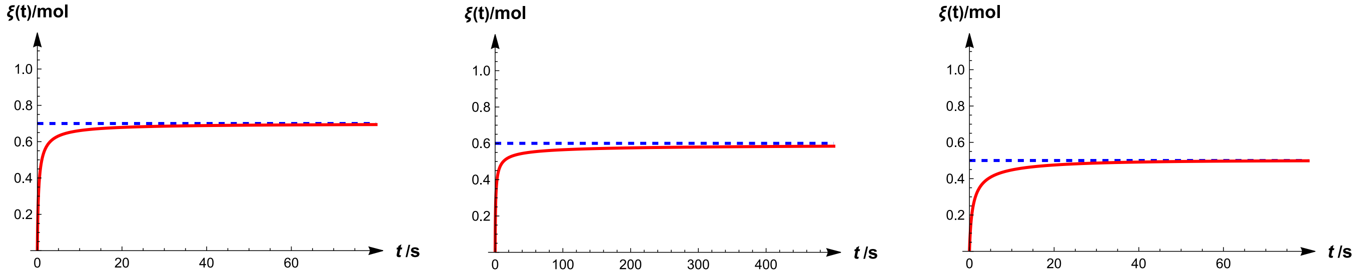

FIG. 3. Reaction extents are tending to different constant values depending on the initial conditions in case of the reaction $2X + Y \xrightarrow{k} 2Z$. In all cases, $V = 1\,\mathrm{dm}^3$, $k = 1.0\,\mathrm{dm}^6\,\mathrm{mol}^{-2}\,\mathrm{s}^{-1}$, and $c_Z^0 = 3$ mol dm^{-3}. Other initial conditions are $c_X^0/2 = 0.7$ mol dm^{-3} $< c_Y^0 = 1.2$ mol dm^{-3} (left), $c_X^0/2 = c_Y^0 = 0.6$ mol dm^{-3} (middle), and $c_X^0/2 = 0.6$ mol dm^{-3} $> c_Y^0 = 0.5$ mol dm^{-3} (right).

TABLE I. Reaction extent for various reaction types.

Step	c^0	$\dot{\xi} =$	$\xi(t) =$	Case
$X \xrightarrow{k} 2X$	0	$k\xi$	0	Eq. (20)
$X \xrightarrow{k} 2X$	1	$Vk(1+\xi/V)$	$V(e^{kt}-1)$	Eq. (21)
$2X \xrightarrow{k} 3X$	1	$Vk(1+\xi/V)^2$	$\frac{Vkt}{1-kt}$	Eq. (21)
$2X + Y \xrightarrow{k} 2Z$		$Vk(c_X^0 - 2\xi/V)^2 *$ $(c_Y^0 - \xi/V)$		Eq. (22)

Example 5.

1. Reaction $X \xrightarrow{k} Y$ with $c_X^0 = 0$ and c_Y^0 arbitrary, illustrates the first case as here $\dot{\xi} = -k\xi, \xi(0) = 0$ implies $\forall t \in \mathbb{R} : \xi(t) = 0$.
2. Reaction $X \xrightarrow{k} 2X$ with $c_X^0 > 0$ is an illustration for the second case
$$\forall t \in \mathbb{R} : \xi(t) = Vc_X^0(e^{kt} - 1),$$
with $\lim_{t\to+\infty} \xi(t) = +\infty$.
3. Reaction $2X \xrightarrow{k} 3X$ with $c_X^0 > 0$ (Example 4) is another illustration for the second case with
$$\forall t \in [0, \frac{1}{kc_X^0}[: \xi(t) = \frac{ktV(c_X^0)^2}{1 - ktc_X^0},$$
with $\lim_{t\to\frac{1}{kc_X^0}} \xi(t) = +\infty$.
4. Reaction $X + Y \xrightarrow{k} 2X$ is an illustration for the third case
$$\forall t \in \mathbb{R} : \xi(t) = \frac{(-1 + e^{kt(c_X^0 + c_Y^0)})Vc_X^0 c_Y^0}{(c_Y^0 + e^{kt(c_X^0+c_Y^0)}c_X^0)},$$
with $\lim_{t\to+\infty} \xi(t) = Vc_Y^0$, if $c_X^0, c_Y^0 \neq 0$. If either $c_X^0 = 0$ or $c_Y^0 = 0$, then $\forall t \in \mathbb{R} : \xi(t) = 0$.
5. The example $X \longrightarrow Y$ with $c_X^0 = 0, c_Y^0 > 0$ shows that a species (here Y) can have the positive concentration for all positive times in a reaction where "none of the steps" can start.

Table I shows a series of examples illustrating different types of single irreversible reaction steps.

Example 6. Here we analyze the last example of Table I. In the case of the reaction $2X + Y \xrightarrow{k} 2Z$ mimicking water formation one has the following quantities: $R := 1, M := 3, X := H_2, Y := O_2, Z := H_2O$. Furthermore,

$$\boldsymbol{\alpha} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \quad \boldsymbol{\gamma} = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}.$$

The initial value problem to describe the time evolution of the reaction extent is

$$\dot{\xi} = Vk\left(\begin{bmatrix} c_X^0 \\ c_Y^0 \\ c_Z^0 \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}\frac{\xi}{V}\right)^{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}} = Vk\left(c_X^0 - 2\frac{\xi}{V}\right)^2\left(c_Y^0 - \frac{\xi}{V}\right), \tag{23}$$

$$\xi(0) = 0.$$

We can provide only the inverse of the solution to Eq. (23). However, one can state that the reaction extent tends strictly monotonically to its limit (independent of the value of the reaction rate coefficient): $\lim_{t\to+\infty} \xi(t) = \min\left\{\frac{Vc_X^0}{2}, Vc_Y^0\right\}$. Different initial conditions lead to different results, see Fig. 3. Obviously, the third point of Theorem 2 is of main practical use here. For this case, one has the following statement.

Corollary 1. *In the third case of Theorem 2, dividing ξ by the initial concentration and scaled by the quantity $-\gamma_m/V$, we obtain a (pure) number tending to 1 as t tends to infinity,*

$$\boxed{\lim_{t\to+\infty} \frac{\xi(t)}{\min\left\{-\frac{Vc_m^0}{\gamma_m}; \gamma_m < 0\right\}} = 1.}$$

We suggest that instead of saying that the scaling factor is some initial concentration as in the third case of Theorem 2, one can equally well say that the divisor is the limiting value of the reaction extent, as it is shown in the cases in Fig. 4: $Vc_X^0/2, Vc_X^0/2 = Vc_Y^0$, and Vc_Y^0, respectively. This result will come in handy below.

Most authors (Dumon *et al.*, 1993; Peckham, 2001; Vandezande *et al.*, 2013; and Moretti, 2015) propose another procedure. In cases

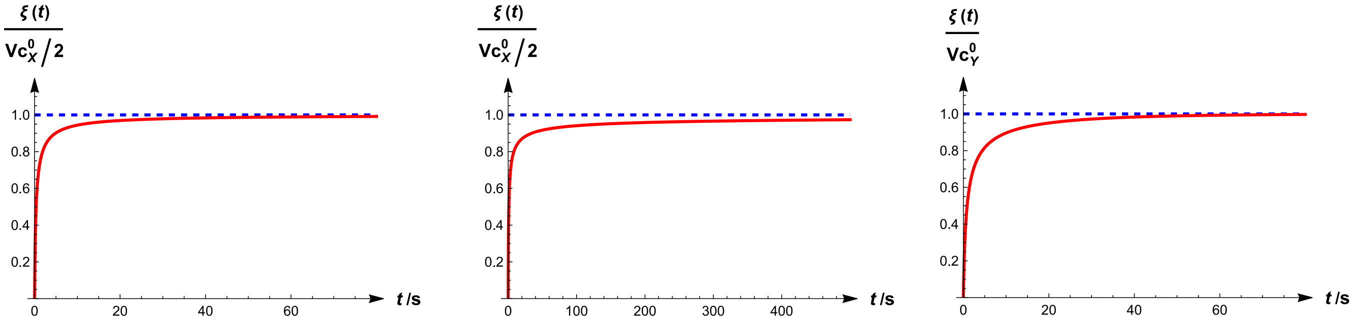

FIG. 4. The scaled reaction extents tend to 1. All data are the same as in the corresponding Fig. 3.

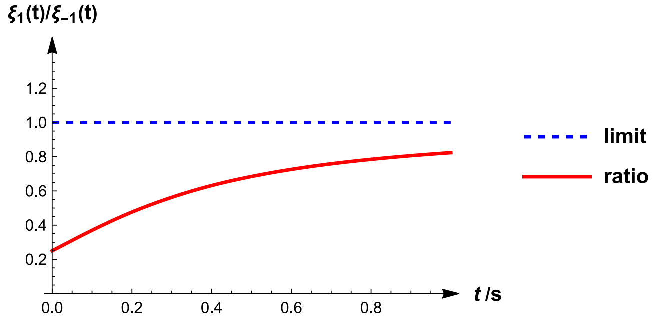

FIG. 5. The ratio $\frac{\xi_1(t)}{\xi_{-1}(t)}$ is tending to 1 from below in case of the reversible reaction $2\mathrm{X} + \mathrm{Y} \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2\mathrm{Z}$ for water formation when at the beginning the forward reaction proceeds slower than the backward reaction. $V = 1\,\mathrm{dm}^3$, $k_1 = 1\,\mathrm{dm}^6\,\mathrm{mol}^{-2}\,\mathrm{s}^{-1}$, $k_{-1} = 1\,\mathrm{dm}^3\,\mathrm{mol}^{-1}\,\mathrm{s}^{-1}$, $c_{\mathrm{X}}^0 = 1/2\,\mathrm{mol\,dm}^{-3}$, $c_{\mathrm{Y}}^0 = 1\,\mathrm{mol\,dm}^{-3}$, and $c_{\mathrm{Z}}^0 = 1\,\mathrm{mol\,dm}^{-3}$.

when $\xi^* := \lim_{t\to+\infty}\xi(t)$ is finite, then $\xi^* = \sup\{\xi(t); t \in \mathbb{R}\}$; thus, ξ^* may be identified (mathematically incorrectly) with $\xi_{\max}$, and surely $\lim_{t\to+\infty}\frac{\xi(t)}{\xi^*} = 1$.

At this point, it may not be obvious how to generalize Corollary 1. In order to treat more complicated cases, we shall choose another way.

B. Detailed balanced reactions

Definition 2. The complex chemical reaction

$$\sum_{m=1}^{M}\alpha_{m,r}\mathrm{X}(m) \underset{k_{-r}}{\overset{k_r}{\rightleftharpoons}} \sum_{m=1}^{M}\beta_{m,r}\mathrm{X}(m) \quad (r = 1, 2, \dots, R), \tag{24}$$

endowed with mass action kinetics is said to be **conditionally detailed balanced** at the positive stationary point $\boldsymbol{c}^*$ if

$$k_r(\boldsymbol{c}^*)^{\boldsymbol{\alpha}_{.,r}} = k_{-r}(\boldsymbol{c}^*)^{\boldsymbol{\beta}_{.,r}} \tag{25}$$

holds **for some** reaction rate coefficients. It is **unconditionally detailed balanced** if Eq. (25) holds **for any** choice of (positive) reaction rate coefficients.

Note that all the steps in Eq. (24) are reversible. In such cases, the forward and backward reaction steps are indexed by r and $-r$, respectively. It is always our choice in which order the forward and backward steps are written, expressing that "forward" and "backward" have no true physical meaning.

1. Ratio of two reaction extents

Suppose we have a reversible reaction,

$$\sum_{m=1}^{M}\alpha_m\mathrm{X}(m) \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} \sum_{m=1}^{M}\beta_m\mathrm{X}(m), \tag{26}$$

being necessarily unconditionally detailed balanced. Then, the initial value problem for the reaction extents is as

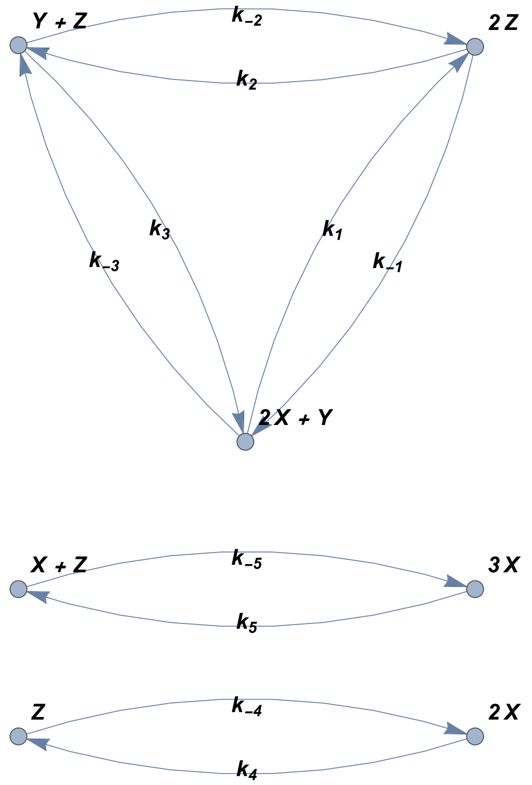

FIG. 6. A triangle coupled with a Wegscheider-type reaction, Wegscheider (1902).

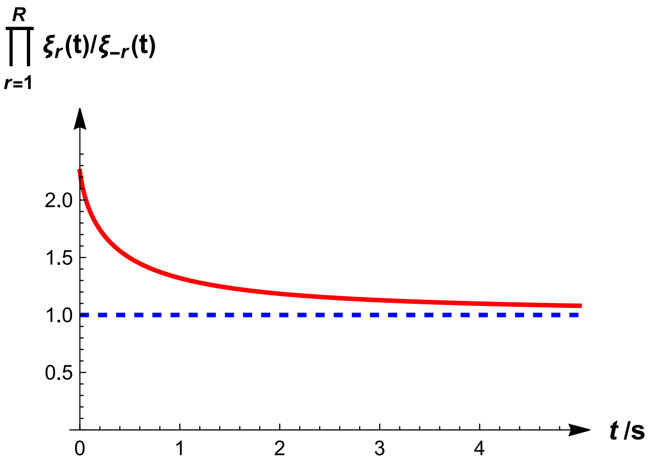

FIG. 7. Convergence of the product of ratios from above in case of reaction in Fig. 6 with parameters implying **detailed balancing** as follows: $k_1 = 1\,\mathrm{dm}^6\,\mathrm{mol}^{-2}\,\mathrm{s}^{-1}$, $k_{-1} = 2\,\mathrm{dm}^3\,\mathrm{mol}^{-1}\,\mathrm{s}^{-1}$, $k_2 = 3\,\mathrm{dm}^3\,\mathrm{mol}^{-1}\,\mathrm{s}^{-1}$, $k_{-2} = 1\,\mathrm{dm}^3\,\mathrm{mol}^{-1}\,\mathrm{s}^{-1}$, $\mathbf{k_3} = \frac{1}{3}\,\mathrm{dm}^3\,\mathrm{mol}^{-1}\,\mathrm{s}^{-1}$, $\mathbf{k_{-3}} = \frac{1}{2}\,\mathrm{dm}^6\,\mathrm{mol}^{-2}\,\mathrm{s}^{-1}$, $k_4 = 3\,\mathrm{dm}^3\,\mathrm{mol}^{-1}\,\mathrm{s}^{-1}$, $k_{-4} = 2\,\mathrm{s}^{-1}$, $k_5 = 3\,\mathrm{dm}^6\,\mathrm{mol}^{-2}\,\mathrm{s}^{-1}$, $k_{-5} = 2\,\mathrm{dm}^3\,\mathrm{mol}^{-1}\,\mathrm{s}^{-1}$, $c_{\mathrm{X}}^0 = c_{\mathrm{Y}}^0 = c_{\mathrm{Z}}^0 = 1\,\mathrm{mol\,dm}^{-3}$, and $V = 1\,\mathrm{dm}^3$.

follows:

$$\dot{\xi}_1 = Vk_1 \prod_{m=1}^{M} (c_m^0 + \gamma_m(\xi_1 - \xi_{-1})/V)^{\alpha_m}, \quad \xi_1(0) = 0,$$

$$\dot{\xi}_{-1} = Vk_{-1} \prod_{m=1}^{M} (c_m^0 + \gamma_m(\xi_1 - \xi_{-1})/V)^{\beta_m}, \quad \xi_{-1}(0) = 0,$$

where $\gamma_m := \beta_m - \alpha_m$.

Proposition 1. *Under the above conditions, one has* $\boxed{\lim_{t\to+\infty}\frac{\xi_1(t)}{\xi_{-1}(t)} = 1.}$

As mentioned before, the proof of this and other statements can be found in the Appendix. Note that initially one only knows that it is the **derivative**s of the reaction extents that have the same value at equilibrium.

Consider the reaction Eq. (26) consisting of a single reversible step.

Conjecture 1. The ratio $\frac{\xi_1(t)}{\xi_{-1}(t)}$ as the function of t is strictly monotonic, except in the case when $k_{-1}(\boldsymbol{c}^0)^{\boldsymbol{\beta}} = k_1(\boldsymbol{c}^0)^{\boldsymbol{\alpha}}$, then it is constant 1.

The meaning of the above conjecture is as follows: if at the beginning, for example, the forward reaction proceeds slower than the backward reaction, then the limit 1 of the ratio is approached from below (see Fig. 5), etc.

Instead of studying other simple reactions, we generalize the above result.

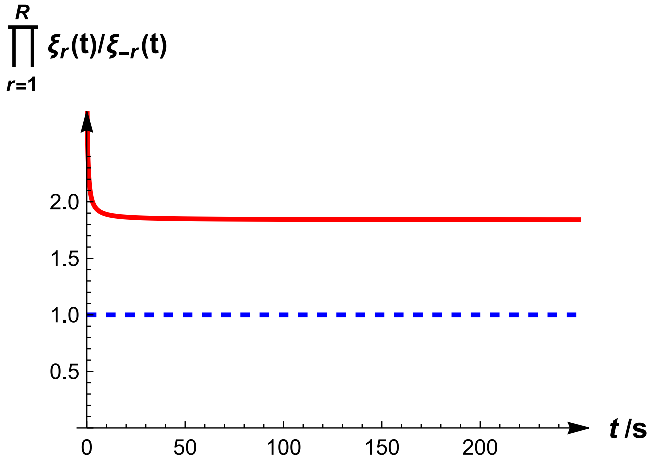

FIG. 8. Convergence of the product of ratios in case of reaction in Fig. 6 with parameters **not fulfilling detailed balance** as follows: $k_1 = 1\,\mathrm{dm}^6\,\mathrm{mol}^{-2}\,\mathrm{s}^{-1}$, $k_{-1} = 2\,\mathrm{dm}^3\,\mathrm{mol}^{-1}\,\mathrm{s}^{-1}$, $k_2 = 3\,\mathrm{dm}^3\,\mathrm{mol}^{-1}\,\mathrm{s}^{-1}$, $k_{-2} = 1\,\mathrm{dm}^3\,\mathrm{mol}^{-1}\,\mathrm{s}^{-1}$, $\mathbf{k_3} = 1\,\mathrm{dm}^3\,\mathrm{mol}^{-1}\,\mathrm{s}^{-1}$, $\mathbf{k_{-3}} = 1\,\mathrm{dm}^6\,\mathrm{mol}^{-2}\,\mathrm{s}^{-1}$, $k_4 = 3\,\mathrm{dm}^3\,\mathrm{mol}^{-1}\,\mathrm{s}^{-1}$, $k_{-4} = 2\,\mathrm{s}^{-1}$, $k_5 = 3\,\mathrm{dm}^6\,\mathrm{mol}^{-2}\,\mathrm{s}^{-1}$, $k_{-5} = 2\,\mathrm{dm}^3\,\mathrm{mol}^{-1}\,\mathrm{s}^{-1}$, $c_X^0 = c_Y^0 = c_Z^0 = 1\,\mathrm{mol}\,\mathrm{dm}^{-3}$, and $V = 1\,\mathrm{dm}^3$.

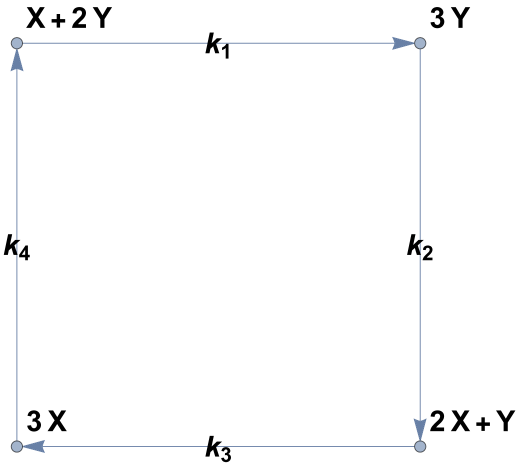

FIG. 9. The Horn–Jackson reaction with $k_1 = k_3 = 1\,\mathrm{dm}^6\,\mathrm{mol}^{-2}\,\mathrm{s}^{-1}$ and $k_2 = k_4 = k$. The value of k is varied as described in the text, its unit is the same as that of the other rate coefficients.

2. Multiplication of the ratios

Theorem 3. *If the complex chemical reaction Eq.* (24) *is conditionally detailed balanced, i.e., for some reaction rate coefficients Eq.* (25) *is fulfilled for each pair of reaction steps, then* $\lim_{t\to+\infty}\prod_{r=1}^{R}\frac{\xi_r(t)}{\xi_{-r}(t)} = 1$.

Proof. It is similar to that of Proposition 1: all the separate factors tend to 1 as $t \to +\infty$. ☐

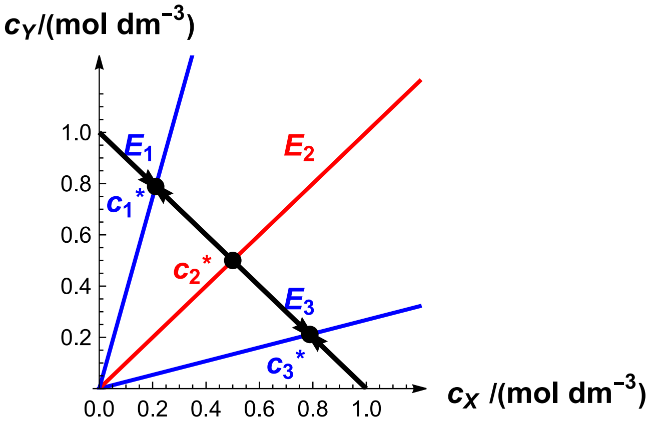

FIG. 10. Phase plane of the reaction network in Fig. 9. The black line is one of the compatibility classes, where the total concentration is unity. E_1, E_2, and E_3 are the sets of stationary points, and their intersections with the line of the given compatibility class result in the actual stationary points $\boldsymbol{c}_1^*$, $\boldsymbol{c}_2^*$, and $\boldsymbol{c}_3^*$. Blue denotes asymptotic stability, and red denotes instability. The black arrows show the direction of the motion along the trajectories in the neighborhood of the attracting stationary points.

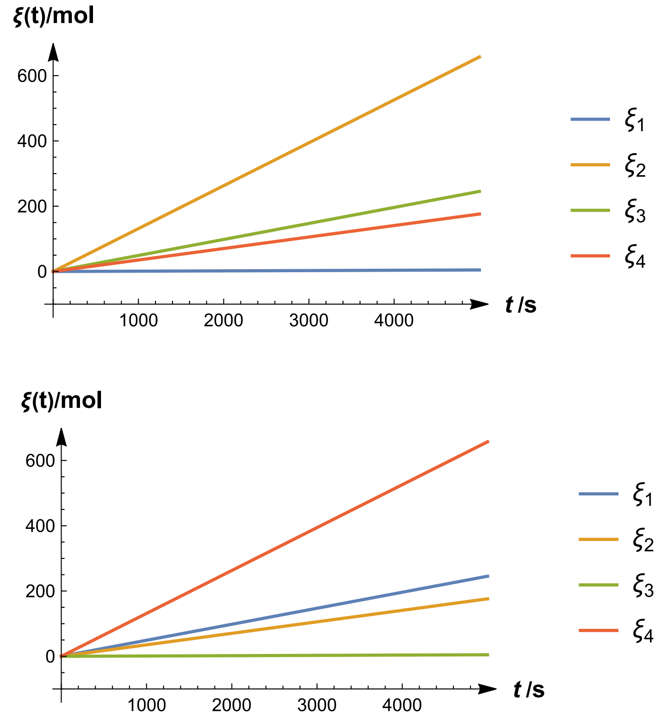

FIG. 11. Reaction extents starting from the attracting domain of $\boldsymbol{c}_1^*$ (above) and $\boldsymbol{c}_3^*$ (below). As there are large differences in the values, the smallest one seems to be zero, but it is not. The reaction steps in Fig. 9 are numbered clockwise, starting with the reaction $\mathrm{X}+2\mathrm{Y} \longrightarrow 3\mathrm{Y}$. Note that the ranking of the reaction extents is different in the two cases.

Example 7. Consider the example in Fig. 6. In this case, the condition of detailed balancing to hold is

$$k_{-1}k_{-2}k_{-3} = k_1k_2k_3, \quad k_3k_5 = k_{-3}k_{-5}, \quad k_3k_4 = k_{-3}k_{-4}$$

as applying either the **circuit conditions** and the **spanning forest conditions** (Feinberg *et al.*, 1989) or use the algebraic condition coming from the **Fredholm alternative theorem** [see Tóth *et al.* (2018, p. 133)] gives. The calculations lead to similar equalities in more complicated cases and can be carried out using the function `DetailedBalanced` of the package `ReactionKinetics`, supplement to the book by Tóth *et al.* (2018). See an example on p. 136 of the cited book. Thus, Fig. 7 shows that the ratio tends to 1.

Example 8. Even if the reversible reaction is **conditionally detailed balanced**, it will not blow up, see Boros and Hofbauer (2020). The stationary point exists and is unique, and the product of the ratios will converge; although the limit will be different from 1, see Fig. 8. Let us use l'Hospital's Rule to show this, in the case of a single factor of the product,

$$\lim_{t\to+\infty}\frac{\xi_r(t)}{\xi_{-r}(t)} = \lim_{t\to+\infty}\frac{\dot{\xi}_r(t)}{\dot{\xi}_{-r}(t)} = \frac{\dot{\xi}_r(+\infty)}{\dot{\xi}_{-r}(+\infty)} = \frac{k_r}{k_{-r}}\frac{(\boldsymbol{c}^*)^{\boldsymbol{\alpha}(.,r)}}{(\boldsymbol{c}^*)^{\boldsymbol{\beta}(.,r)}},$$

which is not necessarily equal to 1.

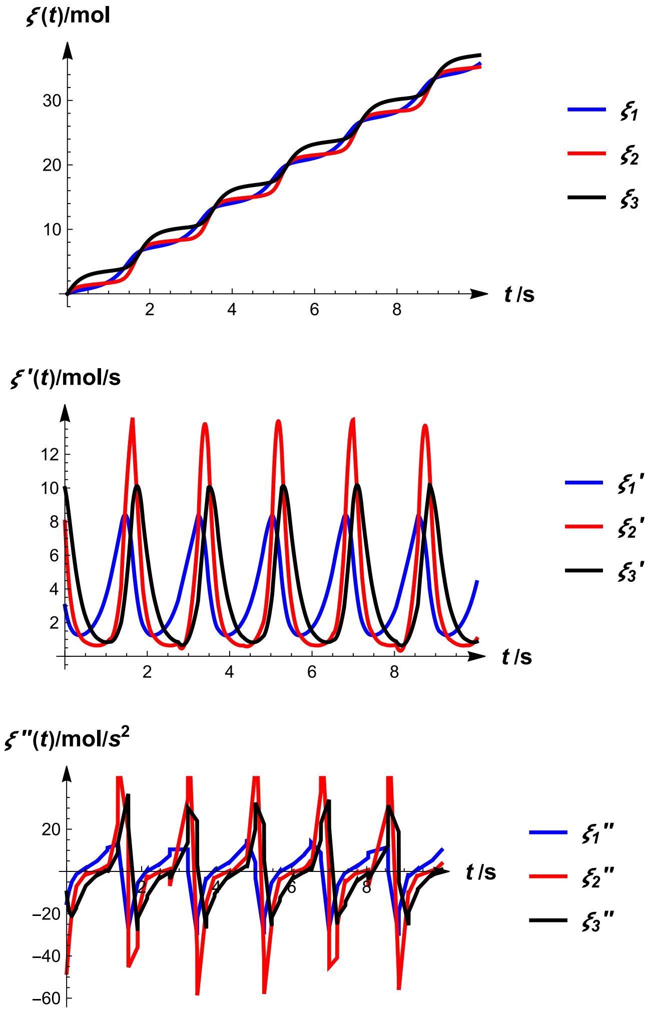

FIG. 12. The individual reaction extents and their first and second derivatives in case of the irreversible Lotka–Volterra reaction with $k_1 = 3\,\mathrm{s}^{-1}$, $k_2 = 4\,\mathrm{dm}^3\ \mathrm{mol}^{-1}\ \mathrm{s}^{-1}$, $k_3 = 5\,\mathrm{s}^{-1}$, $c_\mathrm{X}^0 = 1\ \mathrm{mol\ dm}^{-3}$, $c_\mathrm{Y}^0 = 2\ \mathrm{mol\ dm}^{-3}$, and $V = 1\,\mathrm{dm}^3$.

IV. WHAT IF THE CONDITIONS ARE NOT FULFILLED?

In the first part of the paper, we calculated the reaction extents for the reaction steps of simple reactions, for detailed balanced reactions, etc. Our main question in the present part is: What happens if one takes an exotic reaction that has multiple stationary point(s), and shows oscillations or even chaos?

A. Multistationarity

Horn and Jackson (1972) have shown that the complex chemical reaction in Fig. 9 has three (positive) stationary points in every stoichiometric compatibility class if the numerical value of k lies between 0 and $\frac{1}{6}$. To be more specific, let us choose

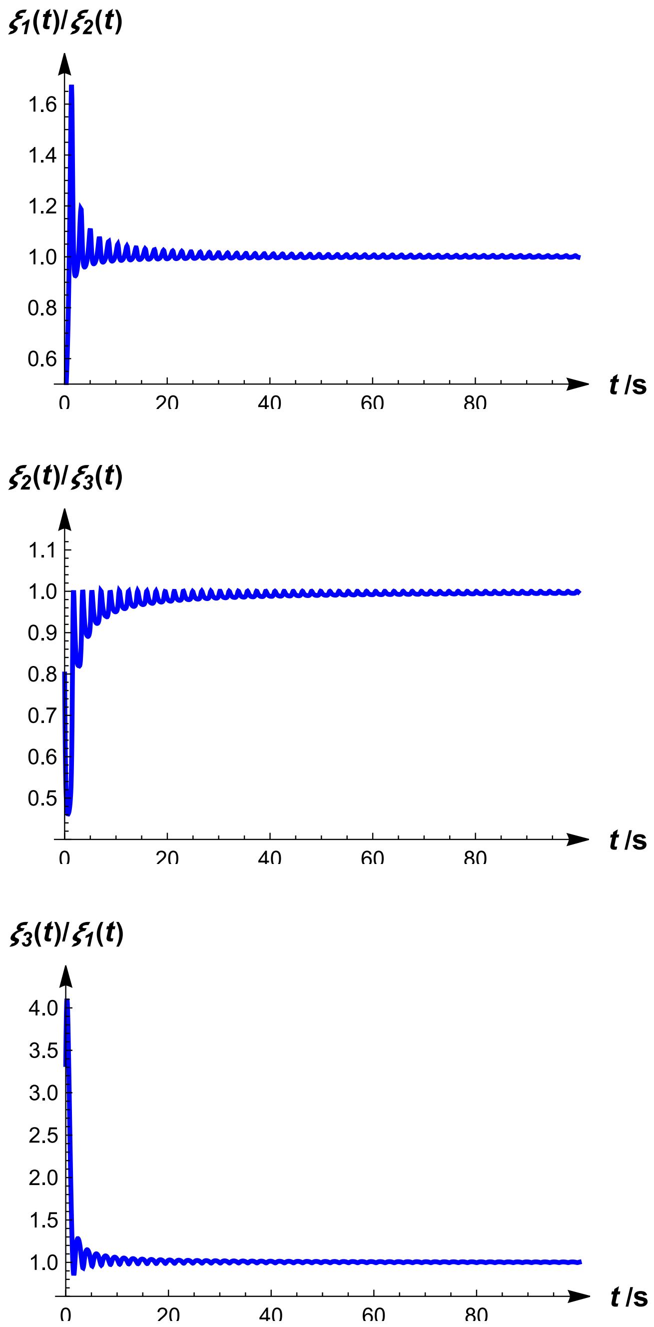

FIG. 13. The ratios of the reaction extents in case of the irreversible Lotka–Volterra reaction with the same parameters as those in Fig. 12.

$k = \frac{1}{10}\,\mathrm{dm}^6\,\mathrm{mol}^{-2}\,\mathrm{s}^{-1}$ and $c_{\mathrm{X}}^0 = c_{\mathrm{Y}}^0 = \frac{1}{2}$ mol dm^{-3}. Then, easy calculation—neglecting the units for simplicity—shows that in the **stoichiometric compatibility** class $\{[c_{\mathrm{X}} \quad c_{\mathrm{Y}}]; c_{\mathrm{X}} + c_{\mathrm{Y}} = 1\}$ (or, for cases when the total concentration is unity) there are three stationary points:

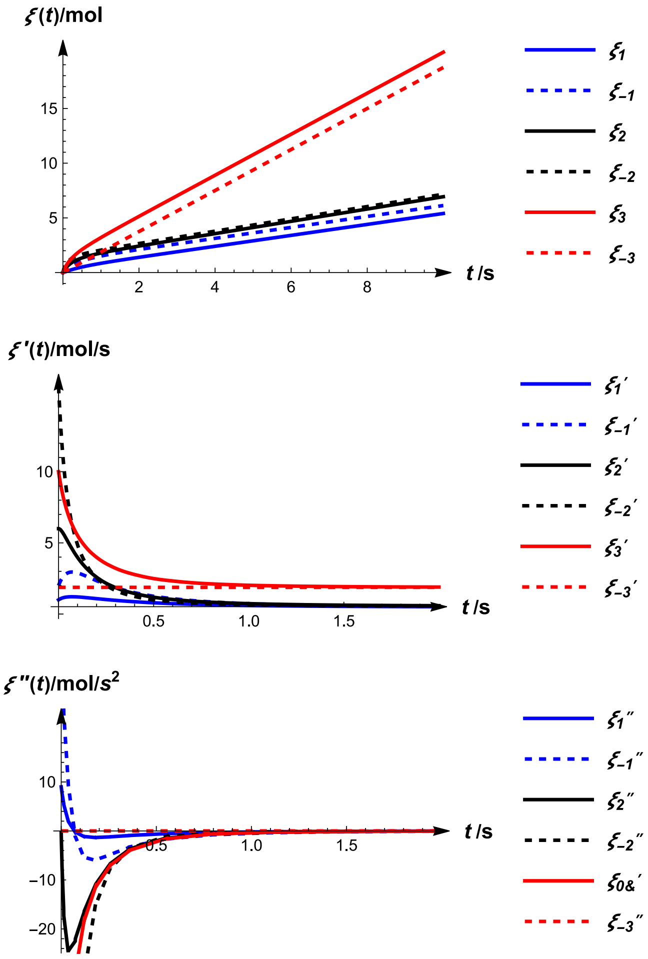

FIG. 14. The individual reaction extents and their first and second derivatives in case of the reversible, conditionally detailed balanced Lotka–Volterra reaction with $k_1 = 1\,\mathrm{s}^{-1}$, $k_{-1} = 2\,\mathrm{dm}^3\,\mathrm{mol}^{-1}\,\mathrm{s}^{-1}$, $k_2 = 3\,\mathrm{dm}^3\,\mathrm{mol}^{-1}\,\mathrm{s}^{-1}$, $k_{-2} = 4\,\mathrm{dm}^3\,\mathrm{mol}^{-1}\,\mathrm{s}^{-1}$, $k_3 = 5\,\mathrm{s}^{-1}$, $k_{-3} = \frac{15}{8}\,\mathrm{mol}\,\mathrm{dm}^{-3}\,\mathrm{s}^{-1}$, $c_{\mathrm{X}}^0 = 1\,\mathrm{mol}\,\mathrm{dm}^{-3}$, $c_{\mathrm{Y}}^0 = 2\,\mathrm{mol}\,\mathrm{dm}^{-3}$, and $V = 1\,\mathrm{dm}^3$.

1. the stationary point $\boldsymbol{c}_1^* := \begin{bmatrix}\frac{1}{3+\sqrt{3}} & \frac{3+\sqrt{3}}{6}\end{bmatrix}$ is asymptotically stable (i.e., attracting) within the given stoichiometric compatibility class with the attracting domain $\{[c_{\mathrm{X}} \quad c_{\mathrm{Y}}] : 0 \le c_{\mathrm{X}} < \frac{1}{3+\sqrt{3}}, c_{\mathrm{X}} + c_{\mathrm{Y}} = 1\}$, and
2. the stationary point $\boldsymbol{c}_2^* := \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\end{bmatrix}$ is unstable (i.e., non-attracting), and
3. the stationary point $\boldsymbol{c}_3^* := \begin{bmatrix}\frac{1}{3-\sqrt{3}} & \frac{3-\sqrt{3}}{6}\end{bmatrix}$ is asymptotically stable (i.e., attracting) within the given stoichiometric compatibility class with the attracting domain $\{[c_{\mathrm{X}} \quad c_{\mathrm{Y}}]; \frac{1}{3-\sqrt{3}} < c_{\mathrm{X}} \le 1, c_{\mathrm{X}} + c_{\mathrm{Y}} = 1\}$.

The character of the stationary points can be determined by using standard linear stability analysis. Figure 10 summarizes the behavior of the trajectories in the neighborhood of the stationary points.

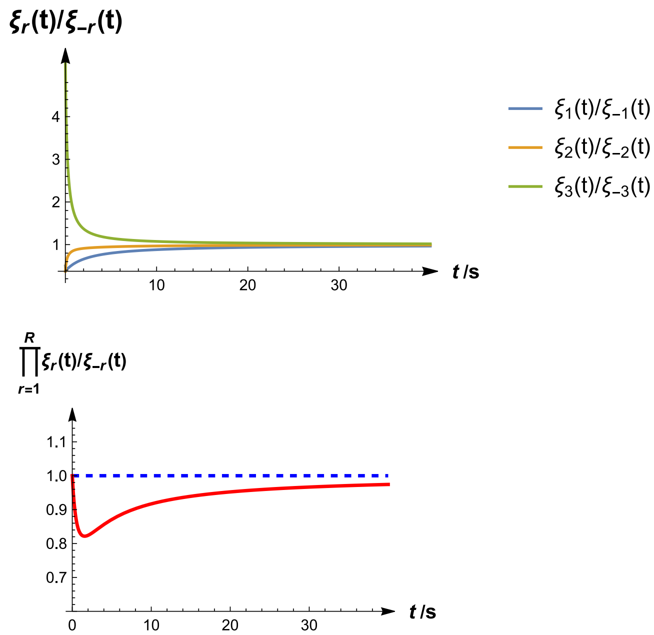

FIG. 15. Top: Time evolution of the ratios of the individual reaction extents in case of the reversible, conditionally detailed balanced Lotka–Volterra reaction with $k_1 = 1\,\mathrm{s}^{-1}$, $k_{-1} = 2\,\mathrm{dm}^3\,\mathrm{mol}^{-1}\,\mathrm{s}^{-1}$, $k_2 = 3\,\mathrm{dm}^3\,\mathrm{mol}^{-1}\,\mathrm{s}^{-1}$, $k_{-2} = 4\,\mathrm{dm}^3\,\mathrm{mol}^{-1}\,\mathrm{s}^{-1}$, $k_3 = 5\,\mathrm{s}^{-1}$, $k_{-3} = \frac{15}{8}\,\mathrm{mol\,dm}^{-3}\,\mathrm{s}^{-1}$, $c_{\mathrm{X}}^0 = 1\,\mathrm{mol\,dm}^{-3}$, $c_{\mathrm{Y}}^0 = 2\,\mathrm{mol\,dm}^{-3}$, and $V = 1\,\mathrm{dm}^3$. Bottom: Time evolution of the product of the ratios is tending to 1.

The reaction extents tend to infinity in all cases, but they are ordered differently for different initial conditions, see Fig. 11. This reflects the fact that not the same reactions are the fastest or slowest in the two cases.

B. Oscillation

We shall study here two oscillatory reactions. First, the often used Lotka–Volterra reaction (Lotka, 1920 and Volterra, 1926) comes that is not only theoretically interesting because it can be used to describe the oscillations in cold flames (Frank-Kamenetskii, 1947 or Frank-Kamenetskii, 2015). The experimentally based reaction by Rábai (1998) aimed at describing pH oscillations follows as the second test problem. One may say that the Brusselator model by Prigogine and Lefever (1968) would be a more realistic choice, as it results in limit cycle solutions. However, it has a third-order step that makes the calculations more tedious. The type of calculations shown below would give almost the same kind of results with the Brusselator, too.

1. The Lotka–Volterra reaction

The irreversible and reversible cases behave in qualitatively different ways.

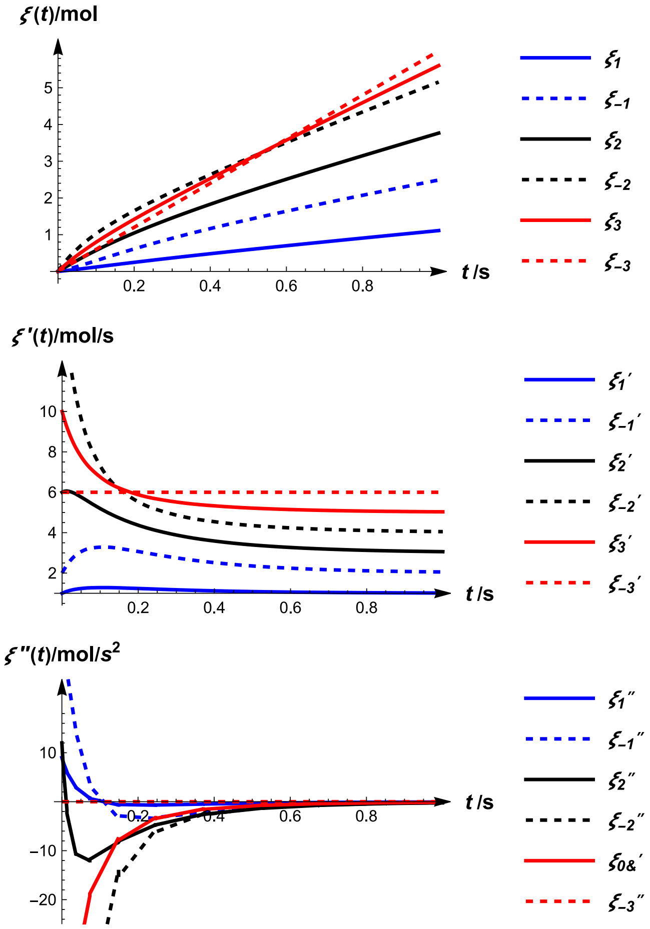

FIG. 16. The individual reaction extents and their derivatives in case of the reversible, not detailed balanced Lotka–Volterra reaction with $k_1 = 1\,\mathrm{s}^{-1}$, $k_{-1} = 2\,\mathrm{dm}^3\,\mathrm{mol}^{-1}\,\mathrm{s}^{-1}$, $k_2 = 3\,\mathrm{dm}^3\,\mathrm{mol}^{-1}\,\mathrm{s}^{-1}$, $k_{-2} = 4\,\mathrm{dm}^3\,\mathrm{mol}^{-1}\,\mathrm{s}^{-1}$, $k_3 = 5\,\mathrm{s}^{-1}$, $k_{-3} = 6\,\mathrm{mol\,dm}^{-3}\,\mathrm{s}^{-1}$, $c_{\mathrm{X}}^0 = 1\,\mathrm{mol\,dm}^{-3}$, $c_{\mathrm{Y}}^0 = 2\,\mathrm{mol\,dm}^{-3}$, and $V = 1\,\mathrm{dm}^3$.

a. Irreversible case: Póta (1983) and Tóth and Schuman (2003) considered two-species reactions with complexes having a length not more than two. They have shown that within this class the only reaction to show oscillations is the (irreversible) Lotka–Volterra reaction $\mathrm{X} \xrightarrow{k_1} 2\mathrm{X},\ \mathrm{X}+\mathrm{Y} \xrightarrow{k_2} 2\mathrm{Y},\ \mathrm{Y} \xrightarrow{k_3} 0$ [cf. also the paper by Tóth and Hárs (1986) and that by Banaji and Boros (2022)]. It has a single positive stationary point that is stable but not attractive. Note that the individual reaction extents are not oscillating; they are "pulsating" while monotonically increasing to infinity. They have an oscillatory derivative, and the zeros of their second derivative clearly show the endpoints of the periods, see Fig. 12. It may be a good idea to calculate any kind of reaction extent **for a period** in case of oscillatory reactions. We are going to study this point later. It is interesting to have a look at the ratios of the reaction extents, as they seem to

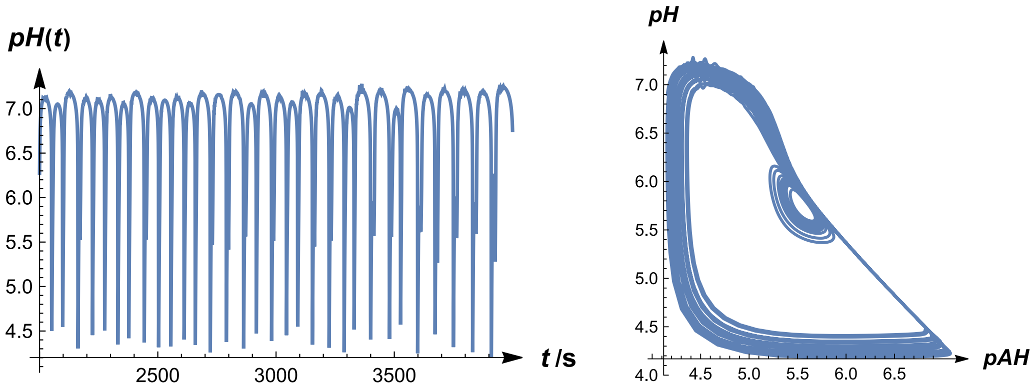

FIG. 17. Time evolution of the pH and the projection of the negative logarithm of the first three coordinates of the trajectory in case of the oscillating Rábai reaction with $k_1 = 10^{10}\ \mathrm{dm^3\ mol^{-1}\ s^{-1}}$, $k_{-1} = 10^3\ \mathrm{s^{-1}}$, $k_2 = 10^6\ \mathrm{dm^3\ mol^{-1}\ s^{-1}}$, $k_3 = 1\ \mathrm{dm^3\ mol^{-1}\ s^{-1}}$, $k_0 = 10^{-3}\ \mathrm{s^{-1}}$, $c^0_{\mathrm{A}^-} = 5 \times 10^{-3}\ \mathrm{mol\ dm^{-3}}$, $c^0_{\mathrm{H}^+} = 10^{-3}\ \mathrm{mol\ dm^{-3}}$, $c^0_{\mathrm{AH}} = 0\ \mathrm{mol\ dm^{-3}}$, $c^0_{\mathrm{P}} = 0\ \mathrm{mol\ dm^{-3}}$, and $V = 1\ \mathrm{dm^3}$.

tend to 1, see Fig. 13. This phenomenon may be related to the fact that the oscillatory solution results in a closed curve in the phase plane of the irreversible Lotka–Volterra reaction.

Consider the irreversible Lotka–Volterra reaction.

Theorem 4. *The limit* $\lim_{t\to+\infty} \frac{\xi_2(t)}{\xi_1(t)}$ *exists and equals 1.*

A similar relationship holds for all the pairs of different indices.

b. Reversible case, conditionally detailed balanced: The reversible Lotka–Volterra reaction $\mathrm{X} \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2\mathrm{X}$, $\mathrm{X} + \mathrm{Y} \underset{k_{-2}}{\overset{k_2}{\rightleftharpoons}} 2\mathrm{Y}$, $\mathrm{Y} \underset{k_{-3}}{\overset{k_3}{\rightleftharpoons}} 0$ is also worth studying. First, let us note that for all values of the reaction rate coefficients, it has a single, positive stationary point. One argument for this is that the reaction steps are reversible; therefore, the system is permanent (Simon, 1995 and Boros, 2019), i.e., the trajectories remain in a compact set. If the trajectories remain in a compact set, then they are either tending to a limit cycle or the stationary point is asymptotically stable. The first possibility is excluded by the above-mentioned theorem by Póta (1983); thus, it is only the second possibility that remains. Figure 14 shows the behavior of the individual reaction extents.

Consider the case when the reaction is conditionally detailed balanced. Equation (25) is fulfilled if and only if

$$k_1 k_2 k_3 = k_{-1} k_{-2} k_{-3} \tag{27}$$

is true. Assuming Eq. (27), Theorem 3 implies that the product of the ratios of the reaction extents tends to 1, see Fig. 15.

c. Reversible case, not detailed balanced: If Condition Eq. (27) does not hold, the reaction still has an **attracting stationary point**. What is more, it has an asymptotically stable stationary point. Figure 16 shows the behavior of the individual reaction extents.

2. The Rábai reaction of pH oscillation

Here, we include a reaction proposed by Rábai (1998) to describe pH oscillations. This reaction has much more direct contact

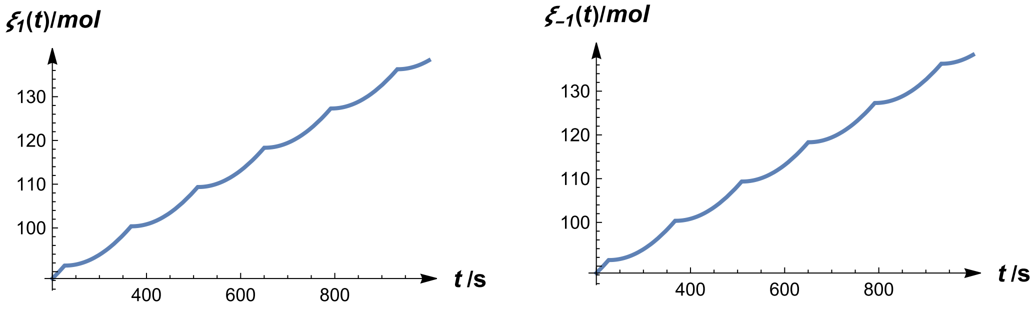

FIG. 18. Reaction extents of the forward (left) and backward (right) reaction steps of the fast equilibrium reaction $\mathrm{A}^- + \mathrm{H}^+ \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} \mathrm{AH}$ of the oscillating Rábai reaction (shown in the same window of time that is presented in Fig. 17).

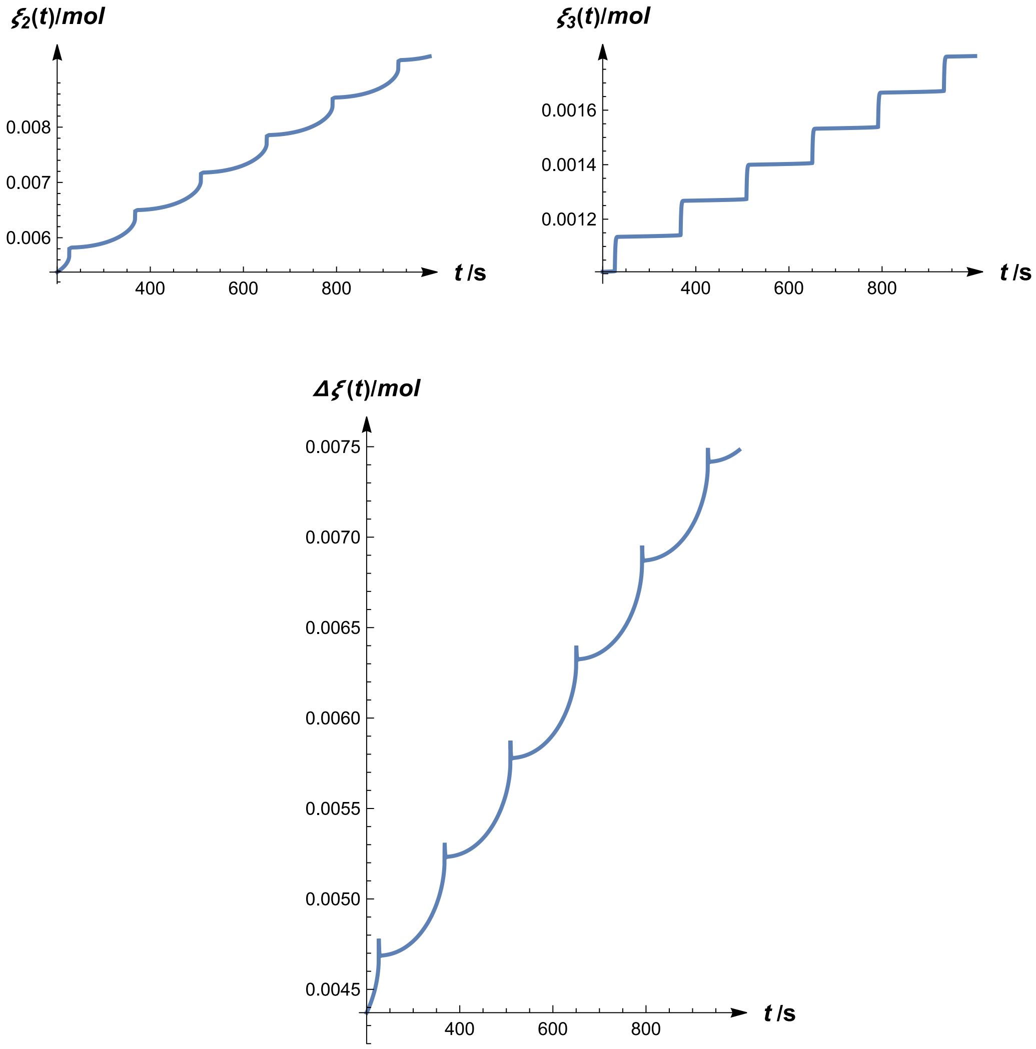

FIG. 19. Reaction extents of the competing reaction steps $\mathrm{AH} + \mathrm{H}^+ + \{\mathrm{B}\} \xrightarrow{k_2} 2\mathrm{H}^+ + \mathrm{P}^-$ (top left) and $\mathrm{H}^+ + \{\mathrm{C}^-\} \xrightarrow{k_3} \mathrm{CH}$ (top right) of the oscillating Rábai reaction shown in the same window of time that is applied in Fig. 17. Interestingly, $\Delta\xi(t) := \xi_3(t) - \xi_2(t)$ is increasing in time but not in a monotonic way (bottom).

with chemical kinetic experiments, and it is much more challenging—from the point of view of numerical mathematics—than the celebrated Lotka–Volterra reaction.

Rábai (1998) starts with the steps

$$\mathrm{A}^- + \mathrm{H}^+ \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} \mathrm{AH},$$

$$\mathrm{AH} + \mathrm{H}^+ + \{\mathrm{B}\} \xrightarrow{k_2} 2\mathrm{H}^+ + \mathrm{P}^-,$$

where {B} is an external species with a constant concentration. This reaction has a single stationary point

$$c^*_{\mathrm{A}^-} = 0, \quad c^*_{\mathrm{H}^+} = c^0_{\mathrm{H}^+} + c^0_{\mathrm{AH}}, \quad c^*_{\mathrm{AH}} = 0, \quad c^*_{\mathrm{P}} = c^0_{\mathrm{A}^-} + c^0_{\mathrm{AH}} + c^0_{\mathrm{P}}$$

specializing into $c^*_{\mathrm{A}^-} = 0, c^*_{\mathrm{H}^+} = c^0_{\mathrm{H}^+}, c^*_{\mathrm{AH}} = 0, c^*_{\mathrm{P}} = c^0_{\mathrm{A}^-}$ with the natural restriction on the initial condition $c^0_{\mathrm{AH}} = 0, c^0_{\mathrm{P}} = 0$.

Putting the reaction into a CSTR (continuously stirred flow-through tank reactor) means in terms of formal reaction kinetics that all the species can flow out and some of the species may flow in so that in the meantime the volume is maintained constant. In the present case, the following steps are added

$$\mathrm{A}^- \xrightarrow{k_0} 0, \tag{28}$$

$$0 \xrightarrow{k_0 c^0_{\mathrm{A}^-}} \mathrm{A}^-, \tag{29}$$

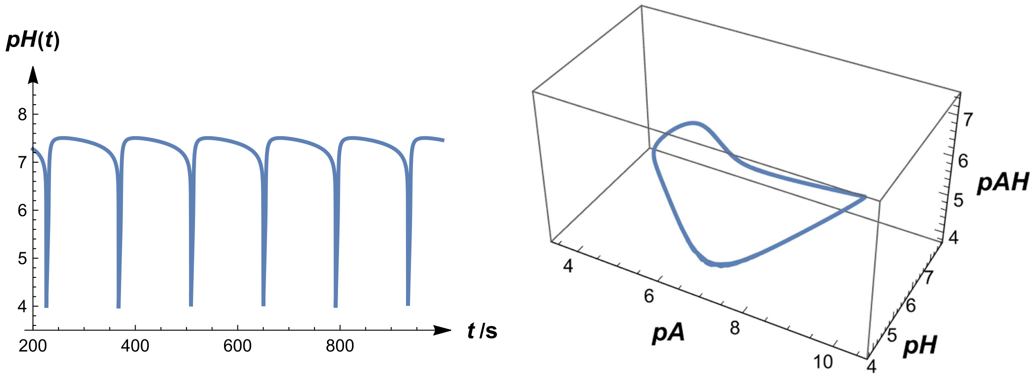

FIG. 20. Time evolution of the pH (left) and the projection of the negative logarithm of the first two coordinates of the trajectory (right) in case of the chaotically oscillating Rábai reaction with the same parameters as in Fig. 17 and with $k_{-3} = 1.5 \times 10^{-2}\,\mathrm{s}^{-1}$ and $k_4 = 5 \times 10^{-2}\,\mathrm{s}^{-1}$.

$$\mathrm{H}^+ \xrightarrow{k_0} 0, \tag{30}$$

$$0 \xrightarrow{k_0 c^0_{\mathrm{H}^+}} \mathrm{H}^+, \tag{31}$$

$$\mathrm{AH} \xrightarrow{k_0} 0, \tag{32}$$

where k_0 is the volumetric flow rate normalized to the volume of the reactor (often called the reciprocal of the residence time) measured in unit s^{-1}. As a result of adding these steps, multistability may occur with appropriately chosen values of the parameters. When the reaction step

$$\mathrm{H}^+ + \{\mathrm{C}^-\} \xrightarrow{k_3} \mathrm{CH} \tag{33}$$

is also added, one may obtain periodic solutions having appropriate parameter values, see Fig. 17. Let us remark that neither the Rábai reaction nor the Lotka–Volterra reaction is mass-conserving.

It is instructive to cast a glance at the reaction extents in such a complex system.

Note that the reaction extents of the fast equilibrium reaction $\mathrm{A}^- + \mathrm{H}^+ \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} \mathrm{AH}$ shown in Fig. 18 are practically the same. Consequently, their ratio tends to 1 as if no other reaction steps were present(!). They are also four–five orders of magnitude larger than those of the autocatalytic production $\mathrm{AH} + \mathrm{H}^+ + \{\mathrm{B}\} \xrightarrow{k_2} 2\mathrm{H}^+ + \mathrm{P}^-$ and the slow pseudo-first-order chemical removal $\mathrm{H}^+ + \{\mathrm{C}^-\} \xrightarrow{k_3} \mathrm{CH}$ of H^+ ion (see Fig. 19). Note the step-wise increase of reaction extent $\xi_3(t)$. Interestingly, $\Delta\xi(t) := \xi_3(t) - \xi_2(t)$ is increasing in time but not in a monotonic way.

C. Chaos

Here, we use a version of the Rábai reaction that **can numerically be shown** to exhibit chaotic dynamics. This is a good model for the experimental pH oscillators also showing behavior that seems to be chaotic according to the usual standards.

When the reaction step Eq. (33) is made reversible,

$$\mathrm{H}^+ + \{\mathrm{C}^-\} \underset{k_{-3}}{\overset{k_3}{\rightleftharpoons}} \mathrm{CH}, \tag{34}$$

and one also introduces both the chemical "removal" and the outflow of CH

$$\mathrm{CH} \xrightarrow{k_4} \mathrm{Q}, \tag{35}$$

$$\mathrm{CH} \xrightarrow{k_0} 0, \tag{36}$$

chaotic solutions are obtained by using appropriate parameters and favorable input concentrations. Figure 20 illustrates this behavior. The reaction extents tend to $+\infty$ (see, for example, Fig. 21) in such a way that their derivative is chaotically oscillating (not shown), as expected. Note that the ratio of the reaction extents $\xi_3(t)/\xi_{-3}(t)$ is not tending to a limit value (1 would be expected for an equilibrium reaction) but it is chaotically oscillating. It indicates that the reversible reaction is chaotically shifted back-and-forth around the equilibrium point. Further studies should explore the dynamics of the reaction extents in such complex systems.

V. CONCLUSIONS

A generalized definition for the reaction extent has been given by Bowen (1968) [included in Chapter 6 of the book edited and partially written by Truesdell (1984)]. Another definition that turned out to be much better fit into the framework of modern formal reaction kinetics in the last 50 years was given by Aris (1965b). Still, neither of them became popular among chemists and chemical engineers. Our goal here was to further generalize the definition (essentially by Aris) to make it compatible with the present theory of reaction kinetics. The results have revealed that there existed a kind of sleeping definition with no use in chemical kinetics, and we show that this should not be the case.

We have introduced the concept of reaction extent for reaction networks of arbitrary complexity (any number of reaction steps and species) without assuming mass action kinetics. The newly defined reaction extent gives the advancement of each individual irreversible

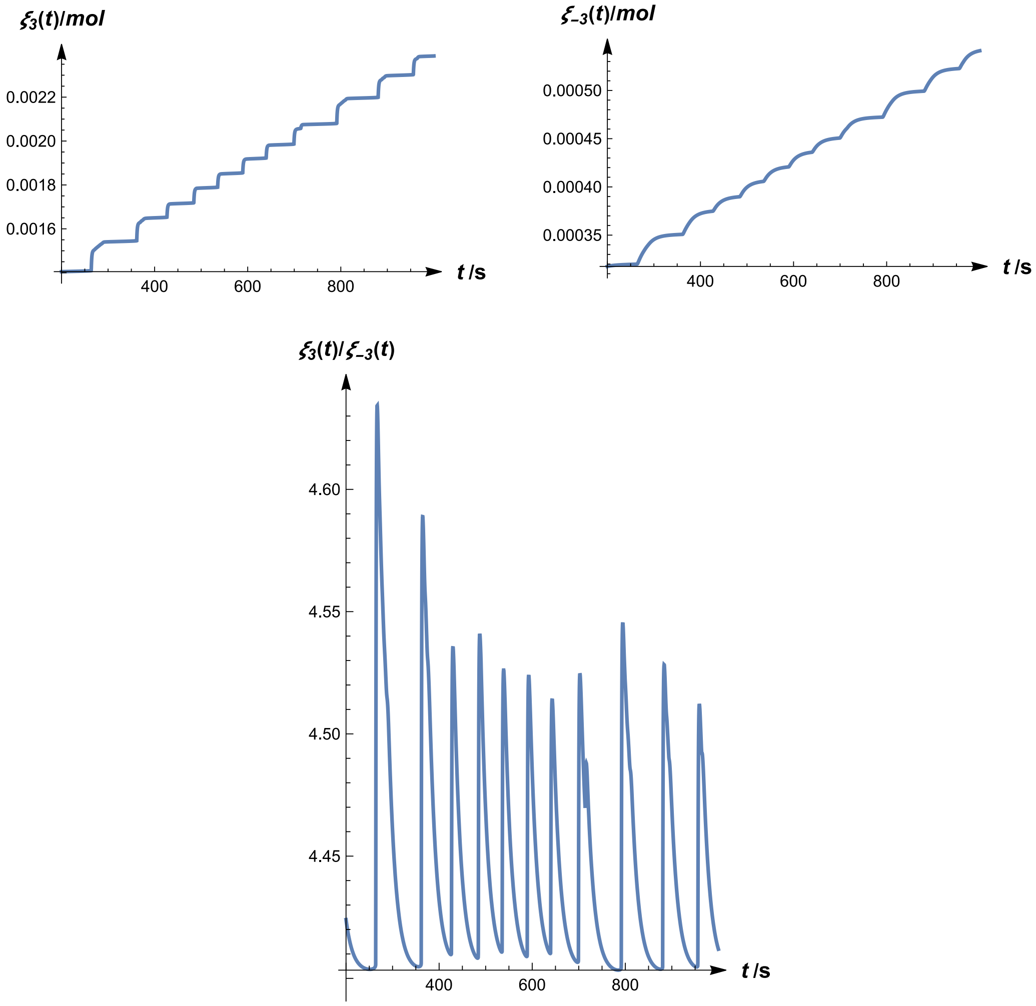

FIG. 21. Time evolution of the reaction extents of the forward (left) and backward (right) steps of reversible reaction Eq. (34) in case of the chaotic Rábai reaction with the same parameters as in Fig. 20. The ratio of the reaction extents $\xi_3(t)/\xi_{-3}(t)$ is also chaotically oscillating (bottom).

reaction step; in the case of reversible reactions, we have a pair of reaction extents. In all the practically important cases, the fact that the reaction extent is strictly monotonically increasing implies that the reaction events never stop. This observation sheds new light on the concept of **dynamic equilibrium**, without alluding to either thermodynamics or statistical mechanics.

After a few statements on the qualitative behavior of the reaction extent, we made efforts to connect the notion with the traditional ones. Thus, we have shown that if the number of reaction steps is one, the reaction extent in the long run (as $t \to +\infty$) tends to 1 if appropriately scaled. We have not used the expression **progress of reaction**, and even less the **reaction coordinate**. We agree that it is convenient to accept the proposal by Aris (1965a) to work with $\frac{1}{V}\xi$, which is usually called the **degree of advancement**. We have also shown that for an arbitrary number of reversible conditionally detailed balanced reaction steps the product of the ratios of the individual reaction extents also tends to 1 as $t \to +\infty$.

One should take into consideration that although in the practically interesting cases, when the number of equations R in Eq. (16) for the reaction extents is larger than M, i.e., the number of the kinetic differential equations in Eq. (10): $R > M$, then the equations for the reaction extents are of a much simpler structure, as the right-hand side of the differential equations, Eq. (16) describing them consist only of a single term. During calculations, we had the experience that it was numerically less demanding to solve the system of differential equations of the reaction extents than those of the concentrations.

The main advantage of the new definition of reaction extent is that by knowing the kinetic model of a reacting system one can now calculate not only the time evolution of the concentration of each reacting species but also the number of occurrences of the individual reaction events.

One can say that the concept of the newly defined reaction extent can be usefully applied to a larger class of reactions than usual, but in some (exotic) cases its use needs further investigation. The interesting results about the peculiar time evolution of reaction extents in the case of oscillating and chaotic chemical systems call for further studies to understand the dynamics of the individual reaction extents in such complex systems. We note that such exploration has not been possible with the simple, standard definition of the reaction extent.

It is for the reader to decide if we succeeded in avoiding all the traps mentioned in the Introduction. Quite a few authors treat the methodology of teaching the concept (Garst, 1974; Vandezande *et al.*, 2013; and Moretti, 2015); we think this approach will only have its *raison d'être* when the scientific background will have been clarified and agreed on.

Let us mention a few limitations and future directions of research. We have assumed throughout that volume (together with temperature and pressure) is constant. Tacitly, we assumed that we deal with homogeneous kinetics; inhomogeneous and heterogeneous systems are not taken into consideration. Also, we have not dealt with reaction-diffusion systems. We mention that recently (Rodrigues *et al.*, 2017 and Pekař, 2022) have applied the concept of reaction extent to the case when diffusion is also present. We have also mentioned a few mathematical conjectures that are challenging to be investigated later.

SUPPLEMENTARY MATERIAL

See the supplementary material for all the calculations and drawings made by using the Wolfram language. Interested readers may request from the authors the .nb file usable for calculations.

ACKNOWLEDGMENTS

The present work has been supported by National Research, Development and Innovation Office, Hungary (No. FK-134332). J.T. is grateful to Dr. J. Karsai (Bolyai Institute, Szeged University) and to Daniel Lichtblau (Wolfram Research) for their help. Members of the Working Committee for Reaction Kinetics and Photochemistry, especially Professor T. Turányi and Professor G. Lente, furthermore Dr. Gy. Póta and Dr. T. Nagy, made a number of useful critical remarks. Highly appreciated are the comments of the anonymous reviewers.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

The authors equally participated in all parts of the paper.

V. Gáspár: Conceptualization (equal); Investigation (equal); Writing – original draft (equal); Writing – review & editing (equal). **J. Tóth:** Conceptualization (equal); Investigation (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available within the article and its supplementary material.

APPENDIX: PROOFS AND NOTATIONS

Proof of Theorem 1.

1. This follows from Eq. (12).
2. As $\boldsymbol{c}$ is the solution of a differential equation with a continuously differentiable right-hand side, it is twice continuously differentiable. The definition Eq. (12) implies $\boldsymbol{\xi} \in \mathcal{C}^2(J, \mathbb{R}^R)$.
3. Take the derivative of Eq. (12) and use the third equation of Eq. (13). The initial condition also comes from Eq. (12).
4. The derivative of ξ_r is non-negative.
5. If the derivative of ξ_r is positive for all $t \in J$, then ξ_r is strictly monotonically increasing. If it is zero at some time $t_0 \in J$, then for some $m : \alpha_{m,r} > 0$ and $c_m(t_0) = 0$. However, $c_m(t_0) = 0$ can only hold if $c_m(0) = 0$ held at the beginning, because an initially positive concentration cannot turn into zero, see Volpert and Hudyaev (1985, Theorem 1, p. 617). But then $c_m(t) = 0$ for all $t \in J$. Thus, in this case, for all $t \in J : \dot{\xi}_r(t) = 0$, this, however, together with the initial condition $\xi_r(0) = 0$ implies that for all $t \in J : \xi_r(t) = 0$; therefore, ξ_r is not strictly monotonically increasing, it is constant zero in this case.

$\square$

Proof of Theorem 2.

1. The condition in Eq. (20) is equivalent to $\forall t \in \mathbb{R}_0^+$ and for all such $m = 1, 2, \ldots, M$ for which $\alpha_m \neq 0$ one has $c_m(t) = 0$; thus, the right-hand side of Eq. (19) is zero. This together with the initial condition $\xi(0) = 0$ proves the statement.
2. The condition in Eq. (21) is equivalent to the right-hand side of Eq. (19) is positive, and

$$\forall t \in \mathrm{Dom}(\xi) :$$

$$\ddot{\xi}(t) = k \sum_{m=1}^{M} \frac{\alpha_m \gamma_m}{c_m^0 + \gamma_m \xi(t)/V} \prod_{p=1}^{M} (c_p^0 + \gamma_p \xi(t)/V)^{\alpha_p} > 0,$$

$$\forall t \in \mathrm{Dom}(\xi) : \ddot{\xi}(t) = \sum_{m=1}^{M} \frac{\alpha_m \gamma_m}{c_m^0 + \gamma_m \xi(t)/V} \frac{\dot{\xi}(t)}{V} > 0,$$

thus, ξ is not only strictly monotonically increasing but also strictly convex from below proving the statement.
3. To calculate a limit when $t \to +\infty$, we show that

$$\sup(\mathrm{Dom}(\boldsymbol{c})) = +\infty \text{ if and only if } \sup(\mathrm{Dom}(\xi)) = +\infty.$$

If all the species are consumed, then the above relation is obviously true. Suppose that not all the species are consumed, e.g., suppose $\mathrm{X}(m)$ is consumed and $\mathrm{X}(p)$ is produced. Then, the induced kinetic differential equation of the reaction Eq. (18)

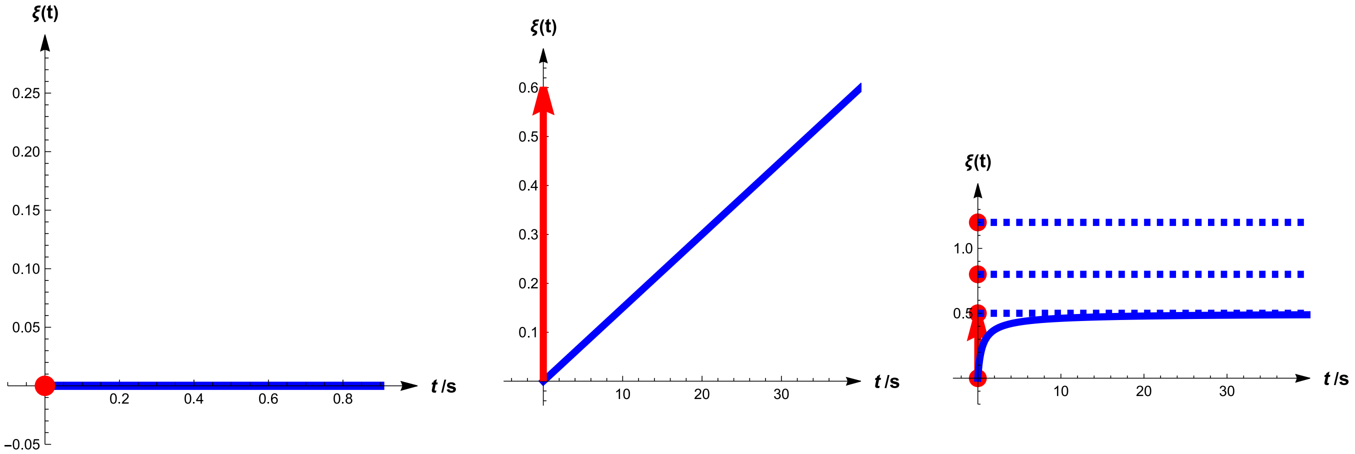

FIG. 22. The different possible structures of the phase line of Eq. (19) are shown. Left: zero is the single stationary point. Middle: there is no stationary point. Right: there are stationary points (at least one; here there are three). The stationary points and trajectories in the phase line representation (vertical axis) are red, while the solutions as a function of time (horizontal axes) are blue. Dashed blue in the right figure denotes the heights corresponding to the stationary points—two of them are never reached from the chosen initial value (0).

TABLE II. Notations I.

Notation	Meaning	Unit	Typical value
$[a, b[$	left-closed, right-open interval		
$\lfloor x \rfloor$	lower integer part of the real number x		
$\Longrightarrow$	implies		
$\in$	belongs to		
$\forall$	universal quantifier		"for all"
$\exists$	existential quantifier		"there is"
$\boldsymbol{A}^\top$	transpose of the matrix $\boldsymbol{A}$		
c_m, c_{X}	concentration of X(m) and X	mol dm^{-3}	
$\boldsymbol{c}$	vector of concentrations		
c_m^0	initial concentration of X(m)	mol dm^{-3}	
c_m^*	stationary concentration of X(m)	mol dm^{-3}	
$\mathcal{C}^i(A, B)$	i times continuously differentiable functions from A into B		
Dom(u)	the domain of the function $t \mapsto u(t)$		
$J \subset \mathbb{R}$	an open interval		
k, k_r, k_{-r}	reaction rate coefficient	$(\text{mol dm}^{-3})^{1-\sum_{m=1}^{M}\alpha_{m,r}}\ \text{s}^{-1}$	
k_0	normalized volumetric flow rate	s^{-1}	
M	the number of chemical species		$\in \mathbb{N}$
n_m	the quantity of species X(m)	mol	$\in \mathbb{N}$
n_m^0	the initial quantity of species X(m)	mol	$\in \mathbb{N}$
$\boldsymbol{n}$	the vector of the quantity of species		
$\boldsymbol{n}^0$	the vector of the initial quantity of species		
$\mathbb{N}$	the set of positive integers		
$\mathbb{N}_0$	the set of non-negative integers		

TABLE III. Notations II.

Notation	Meaning	Unit	Typical value
$rate_r$	rate of the *r*th reaction step	mol dm^{-3} s^{-1}	
$\mathbf{rate}$	vector of reaction rates		
$\mathbb{R}$	the set of real numbers		
$\mathbb{R}^+$	the set of positive real numbers		
$\mathbb{R}^+_0$	the set of non-negative real numbers		
t	time	s	$\in \mathbb{R}$
V	volume	dm^3	
W_r	number of occurrences of the *r*th step		$\in \mathbb{N}_0$
$\boldsymbol{W}$	vector of number of occurrences		
x_m	*m*th dependent variable in a differential equation		
$\boldsymbol{x}$	vector of variables in a differential equation		
X, Y, X(m)	chemical species		
$\alpha_{m,r}, \alpha_m$	stoichiometric coefficient in the reactant complex	1	0, 1, 2, 3
$\boldsymbol{\alpha}$	matrix of reactant complex vectors		
$\beta_{m,r}, \beta_m$	stoichiometric coefficient in the product complex	1	0, 1, 2, 3
$\boldsymbol{\beta}$	matrix of product complex vectors		
$\gamma_{m,r}, \gamma_m$	stoichiometric number	1	$-3, -2, \ldots, 2, 3$
$\boldsymbol{\gamma}$	stoichiometric matrix		
ξ_r	reaction extent of the *r*th step	mol	
$\boldsymbol{\xi}$	vector of reaction extents		

implies that

$$\frac{\dot{c}_m}{-\gamma_m} + \frac{\dot{c}_p}{\gamma_p} = 0,$$

therefore,

$$-\frac{c_m(t)}{\gamma_m} + \frac{c_p(t)}{\gamma_p} = -\frac{c_m(0)}{\gamma_m} + \frac{c_p(0)}{\gamma_p} > 0,$$

and this together with the positivity of the concentrations implies the statement about the equality of the domains. If X(m) is consumed, then $\gamma_m < 0$; therefore, the factor $\left(c^0_m + \gamma_m \frac{\xi}{V}\right)^{\alpha_m}$ in the right-hand side of Eq. (19) is zero for $\xi = -\frac{Vc^0_m}{\gamma_m}$. Since the solution of Eq. (19) starts from zero and it is strictly monotonically increasing; therefore, it should tend to the smallest zero locus of the right-hand side proving the third part of the statement.

The proof can be considered as an application of phase-line analysis as visualized in Fig. 22. The phase space of Eq. (19) is the half line of non-negative real numbers. The following cases can occur:

1. The origin is a stationary point, then the only solution is constant zero. It will turn out later that this case is less irrelevant from the chemical point of view when one has more than one reaction step.
2. There is no stationary point. Then, the derivative of the reaction extent is always positive; thus, it is strictly monotonically increasing. Obviously, it cannot have a finite limit (it cannot stop), because if it would then the limit would be a stationary point, a contradiction.
3. There are stationary points (one or more). Then, the solution to Eq. (19)—as it starts from the origin and is strictly monotonically increasing—will have as its limit the smallest stationary points as it has been said in the theorem.

□

Proof of Proposition 1.

First, let us note that the concentrations (and, therefore, the reaction extents) do not blow up (Boros and Hofbauer, 2020), i.e., $\sup(\mathrm{Dom}(\boldsymbol{c})) = +\infty$. Next, as

$$\lim_{t\to+\infty} \xi_1(t) = +\infty,$$

$$\lim_{t\to+\infty} \xi_{-1}(t) = +\infty,$$

$$\lim_{t\to+\infty} \dot{\xi}_1(t) = Vk_1(\boldsymbol{c}^*)^{\boldsymbol{\alpha}},$$

$$\lim_{t\to+\infty} \dot{\xi}_{-1}(t) = Vk_{-1}(\boldsymbol{c}^*)^{\boldsymbol{\beta}} = Vk_1(\boldsymbol{c}^*)^{\boldsymbol{\alpha}},$$

one can apply l'Hospital's Rule to get the desired result. □

Proof of Theorem 4.

Let $\omega := \lfloor t/T \rfloor$. Then

$$\begin{aligned}\lim_{t\to+\infty}\frac{\xi_2(t)}{\xi_1(t)} &= \lim_{t\to+\infty}\frac{0+\int_0^t k_2x(s)y(s)\,\mathrm{d}s}{0+\int_0^t k_1x(s)\,\mathrm{d}s}\\ &= \frac{k_2}{k_1}\lim_{t\to+\infty}\left(\frac{\omega\int_0^T x(s)y(s)\,\mathrm{d}s+\int_0^{t-\omega T}x(s)y(s)\,\mathrm{d}s}{\omega\int_0^T x(s)\,\mathrm{d}s+\int_0^{t-\omega T}x(s)\,\mathrm{d}s}\right)\\ &= \frac{k_2\int_0^T x(s)y(s)\,\mathrm{d}s}{k_1\int_0^T x(s)\,\mathrm{d}s}.\end{aligned}$$

In order to prove the statement, it is enough to show that $\int_0^T(k_2x(s)y(s)-k_1x(s))\,\mathrm{d}s=0$. However, using the Newton–Leibniz Theorem, and the fact that $\xi_1(T)=\xi_2(T)$, where T is the period, we have

$$\begin{aligned}\int_0^T(k_2x(s)y(s)-k_1x(s))\,\mathrm{d}s &= \int_0^T(\dot{\xi}_2(s)-\dot{\xi}_1(s))\,\mathrm{d}s\\ &= \xi_2(T)-\xi_1(T)-(\xi_2(0)-\xi_1(0))=0.\end{aligned}$$

□

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